Grover's algorithm with faulty and non-faulty marked items

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Why Grover's search ?

- Grover's algorithm is a quantum search algorithm solving the unstructured search problem in $O(\sqrt{N})$ steps, while any classical algorithm needs $\mathrm{O}(\mathrm{N})$ steps.

L. Grover. A fast quantum mechanical algorithm for database search.
Proceedings of the 28th ACM STOC, 212-219, 1996.

- Grover's algorithm and its generalizations (e.g. amplitude amplification) serve as a building blocks for many other quantum algorithms.

- Simple algorithm with simple analysis

The unstructured search problem

- A search space of N elements

- Some elements have a certain property. Marked elements

- A function to check whether an element is marked:

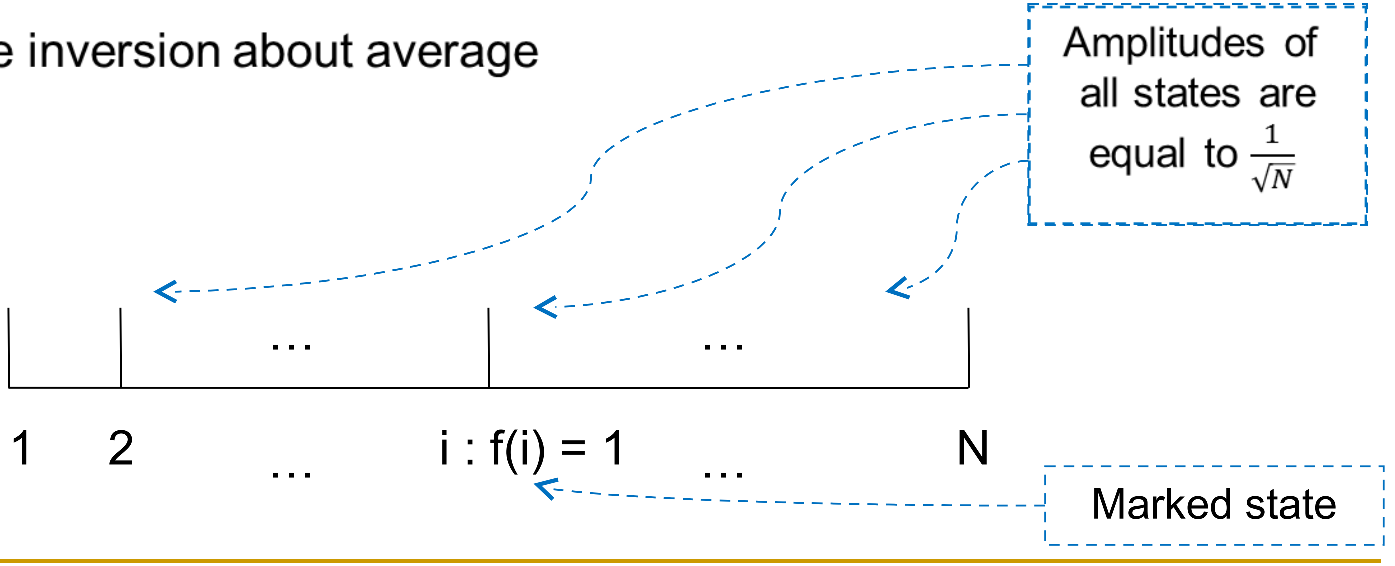
$$f(x) : \{1 \ldots N\} \rightarrow \{0,1\}$$

Oracle

- The unstructured search problem is to find one of marked elements or to conclude that no such element exists.

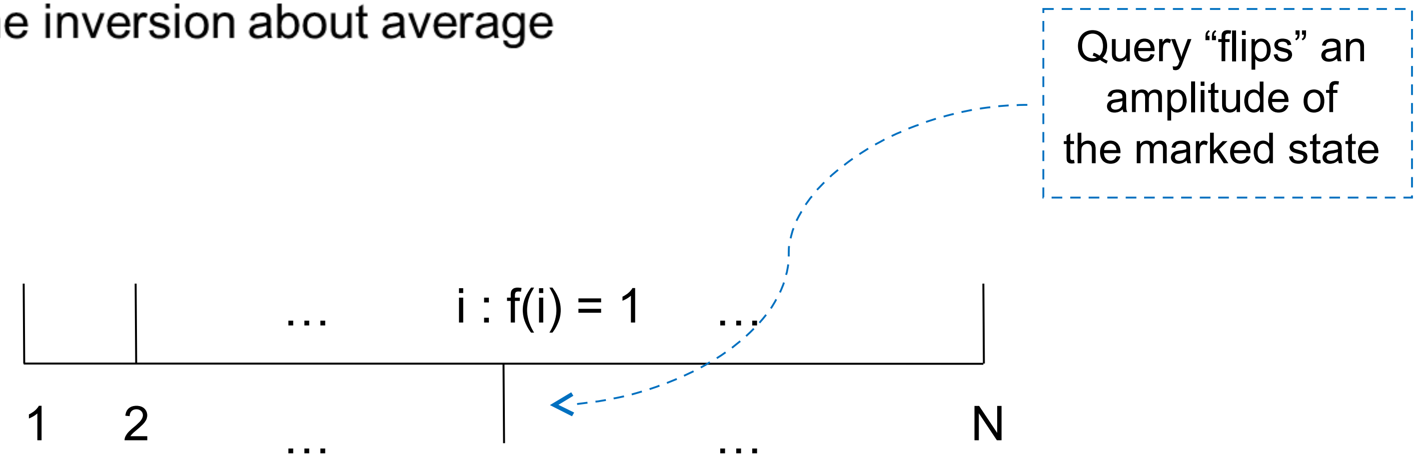
Grover's algorithm

- **Start in equal element superposition**
- Repeat $O(\sqrt{N})$ times
  - Perform query
  - Apply the inversion about average



Grover's algorithm

- Start it equal element superposition
- Repeat $O(\sqrt{N})$ times
  - **Perform query** $Q: |i\rangle \mapsto (-1)^{f(i)}|i\rangle$
  - Apply the inversion about average



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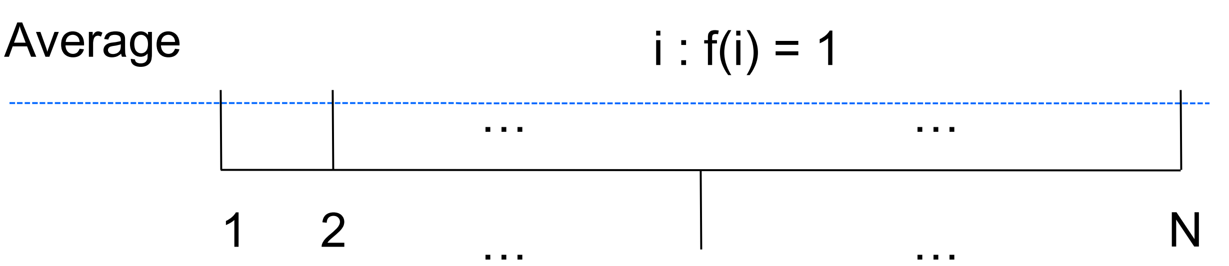

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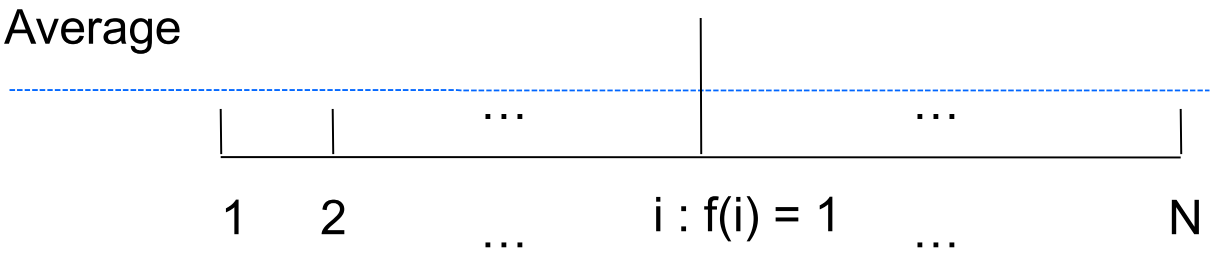
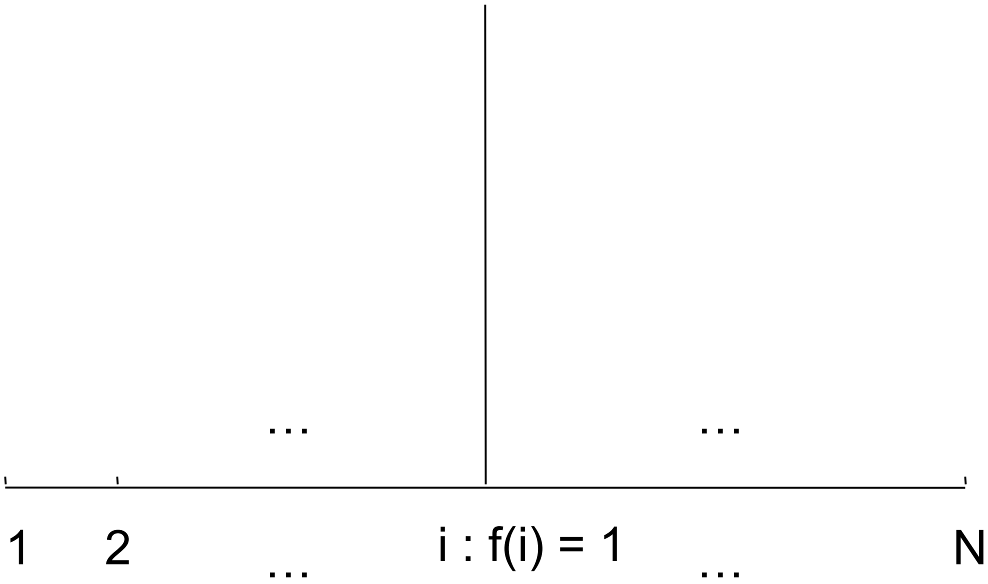


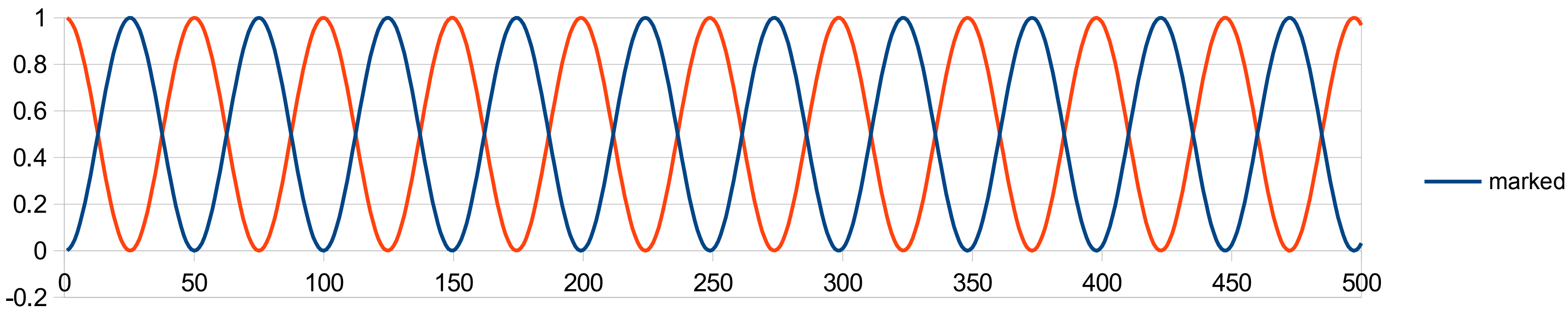
Grover's algorithm

- After $O(\sqrt{N})$ steps probability to the marked element becomes close to 1.



Grover’s algorithm

- Probability to find one of marked elements periodically changes between 0 to 1.



- Number of steps of the algorithm $\frac{\pi}{4}\sqrt{\frac{N}{k}}$.
- Probability to find each marked element is $\frac{1}{k}$.

Number of marked elements

p-faulty unstructured search problem

- A search space of N elements
- Faulty Oracle:
  - With probability $1-p$ performs a query
  - With probability p does nothing

Classically this results in constant slowdown.

Any quantum algorithm that solves the p-faulty unstructured search problem must use $T > \frac{p}{10(1-p)}N$ queries.

O. Regev, L. Schiff. Impossibility of a Quantum Speed-up with a Faulty Oracle. *Proceedings of ICALP'08*, 773-781, 2008.

Research problem

- What if some marked elements are still reported as marked ?

- A search space of N elements
  - $N - k$ elements are non-marked
  - k elements $i_1, \dots, i_k$ are marked
    - $p_j = 0$ – non-faulty marked element
    - $p_j \neq 0$ – faulty marked element

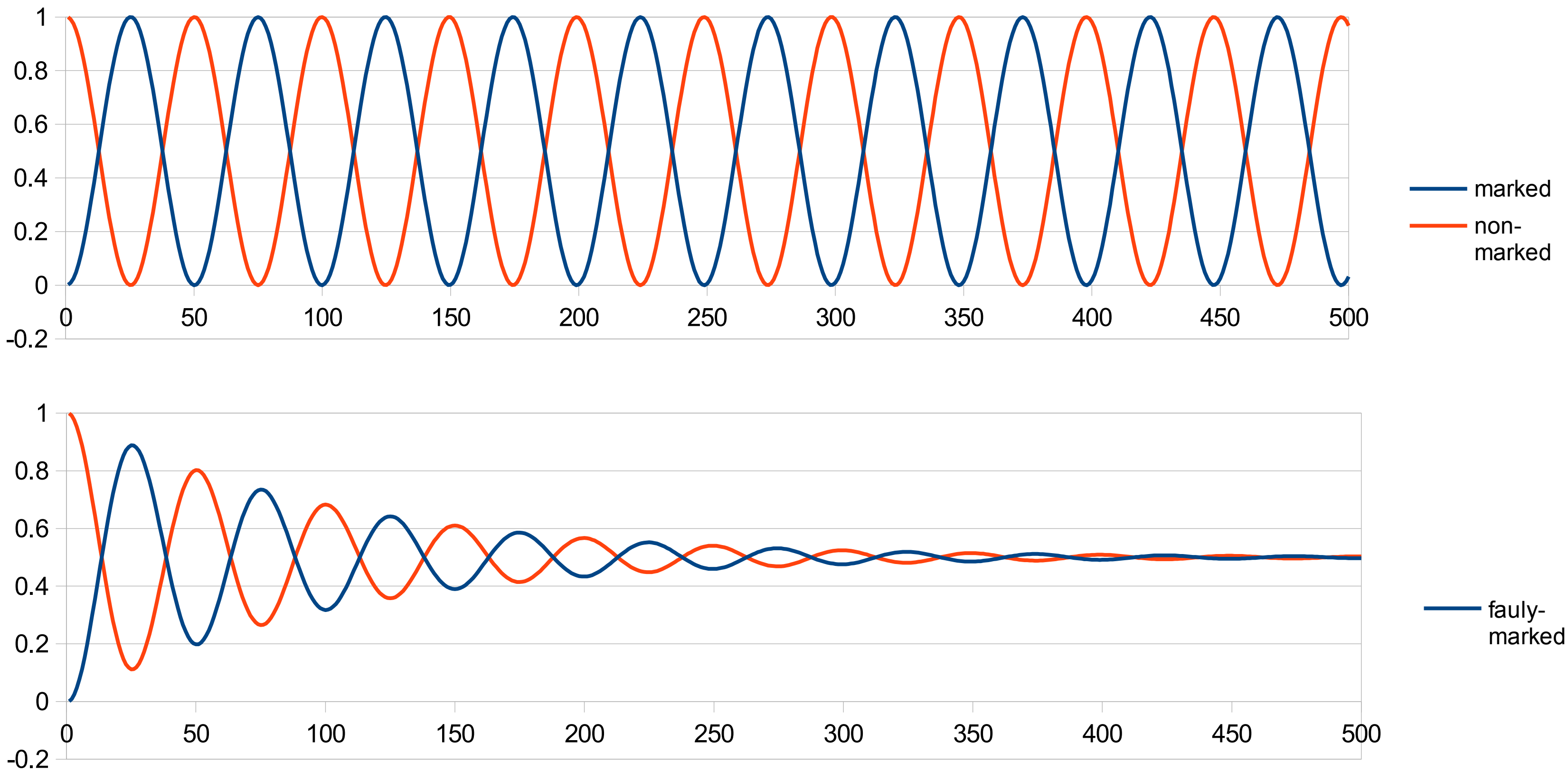
- Each marked element i_j with probability p_j is reported as non-marked.

- Will faulty marked elements stop the algorithm from finding one of non-faulty marked elements ?

Classically the answer is trivial - NO.

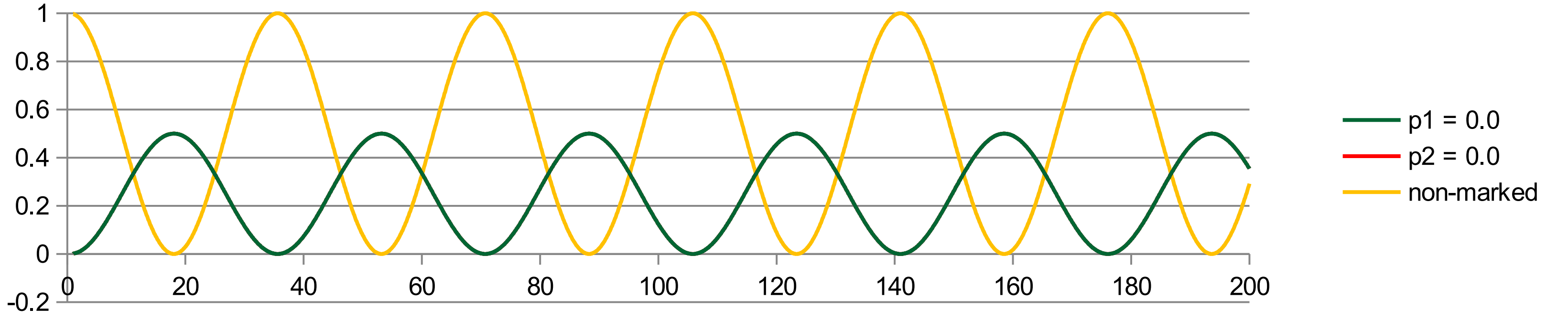
How errors affect the algorithm ?

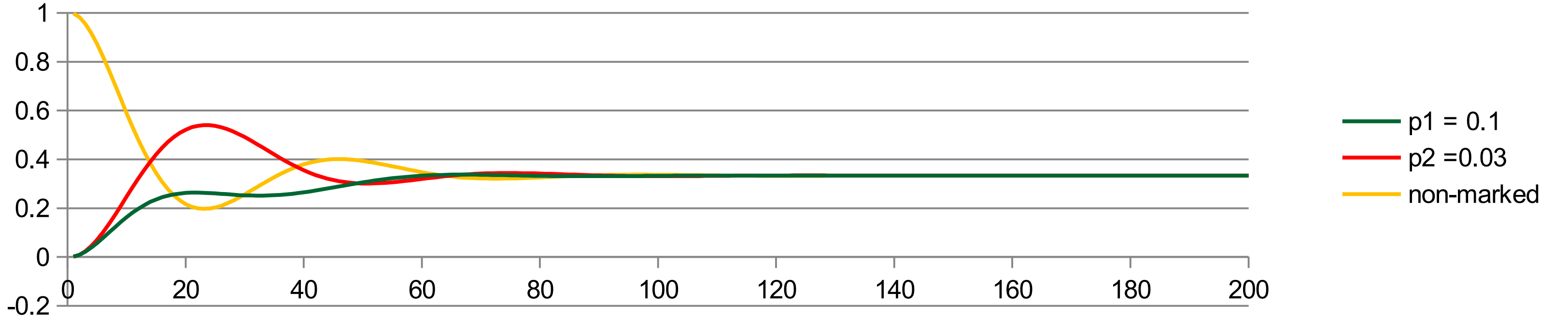
- One marked element



How errors affect the algorithm ?

- Multiple marked elements





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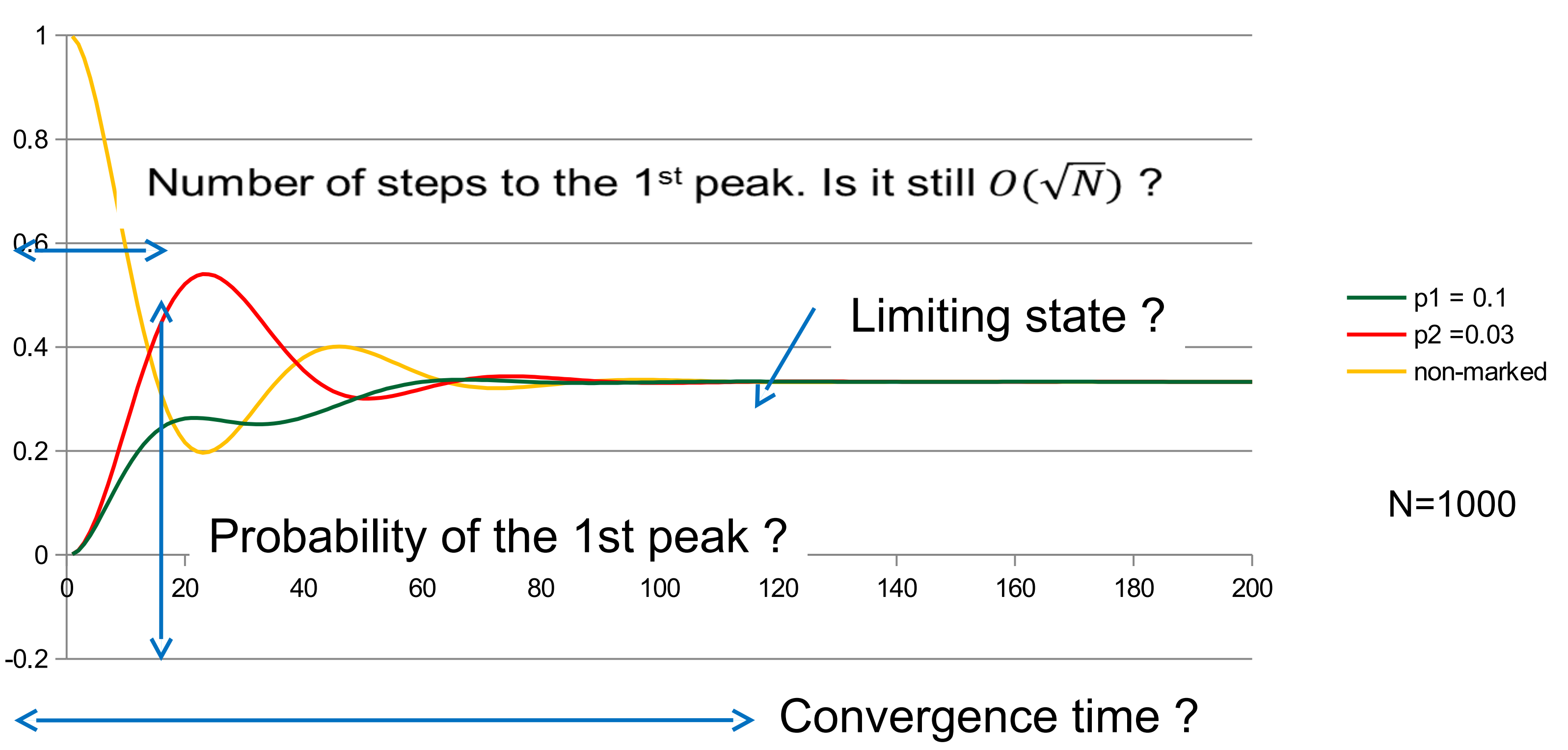


Limiting state

- Convergence time is $O(N)$.

- All $p_j \neq 0$

$$\rho_{\lim} = \frac{1}{k+1}\sum_{j=1}^{k} |i_j\rangle\langle i_j| + \frac{1}{k+1}|\psi_-\rangle\langle\psi_-|$$

Probability to measure one of marked elements is $\frac{k}{k+1}$

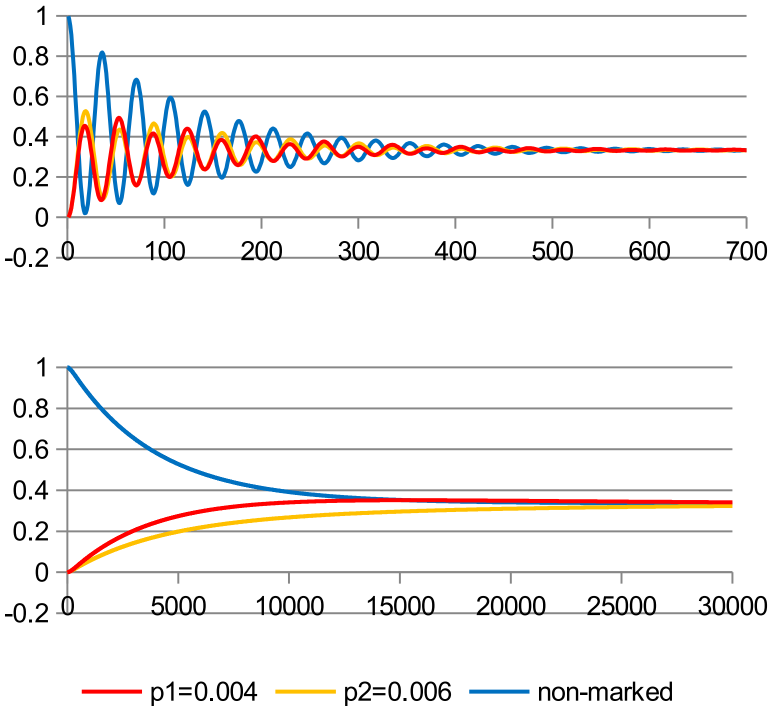
- Some of $p_j = 0$

$$\rho_{\lim} = \frac{1}{m+2}\sum_{j=1}^{m} |i_j\rangle\langle i_j| + \frac{1}{m+2}|\psi_+\rangle\langle\psi_+| + \frac{1}{m+2}|\psi_-\rangle\langle\psi_-|$$

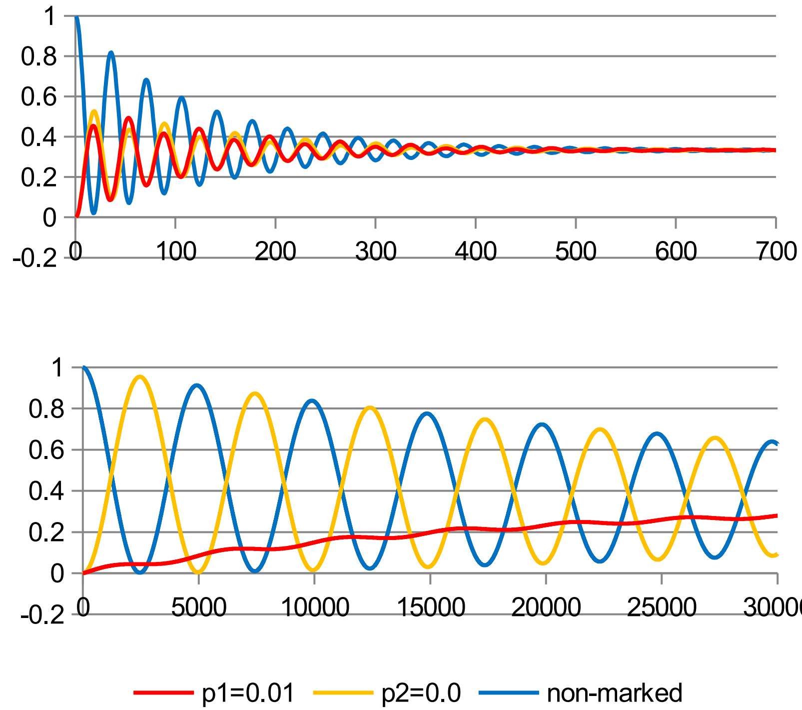
Probability to measure one of marked elements is $\frac{m+1}{m+2}$

First peak

- All $p_j \neq 0$



- Start in equal element superposition



First peak

- All $p_j \neq 0$

  No speed-up over classical exhaustive search is possible

  $O(N)$ time is needed

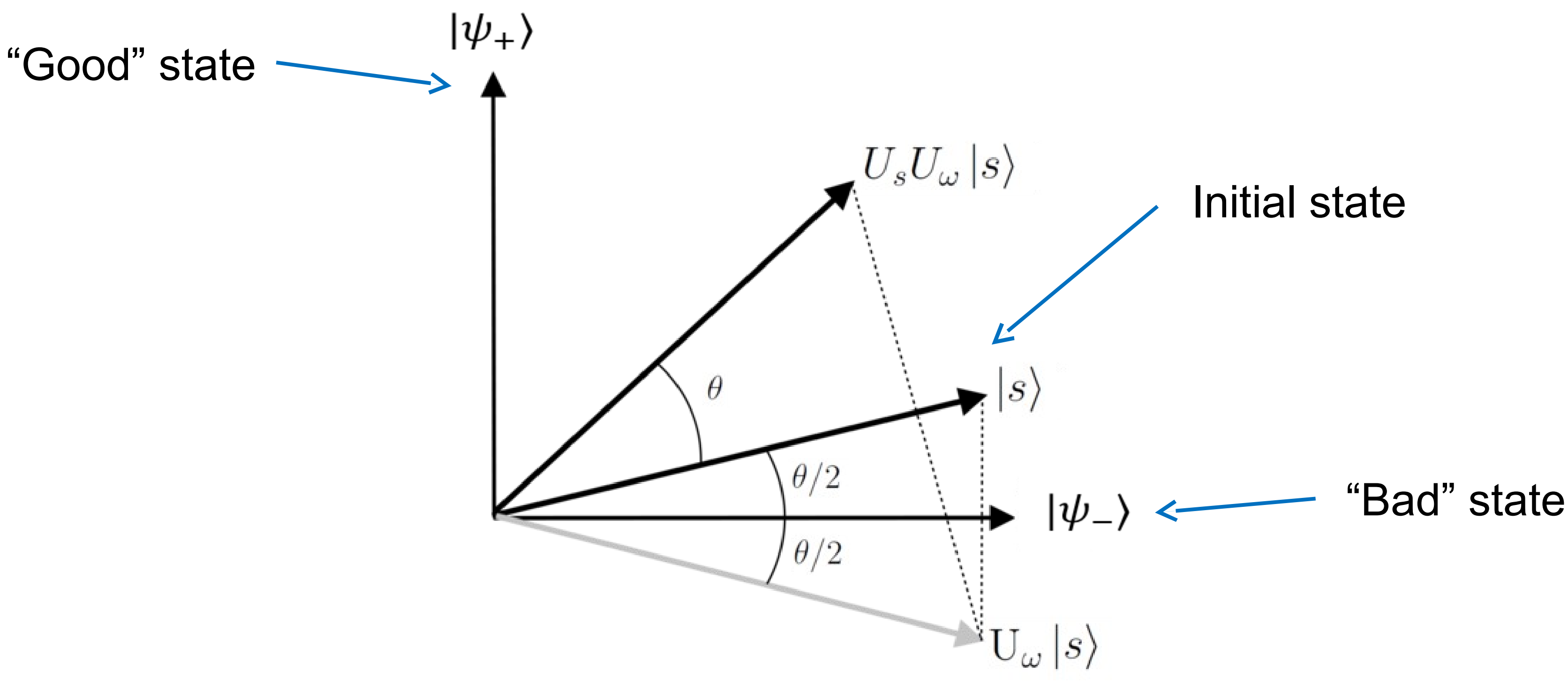
- Some of $p_j = 0$

  Behaves as a Grover's search with $\{p_j : p_j = 0\}$ marked elements

  $O(\sqrt{N})$ time is still possible !

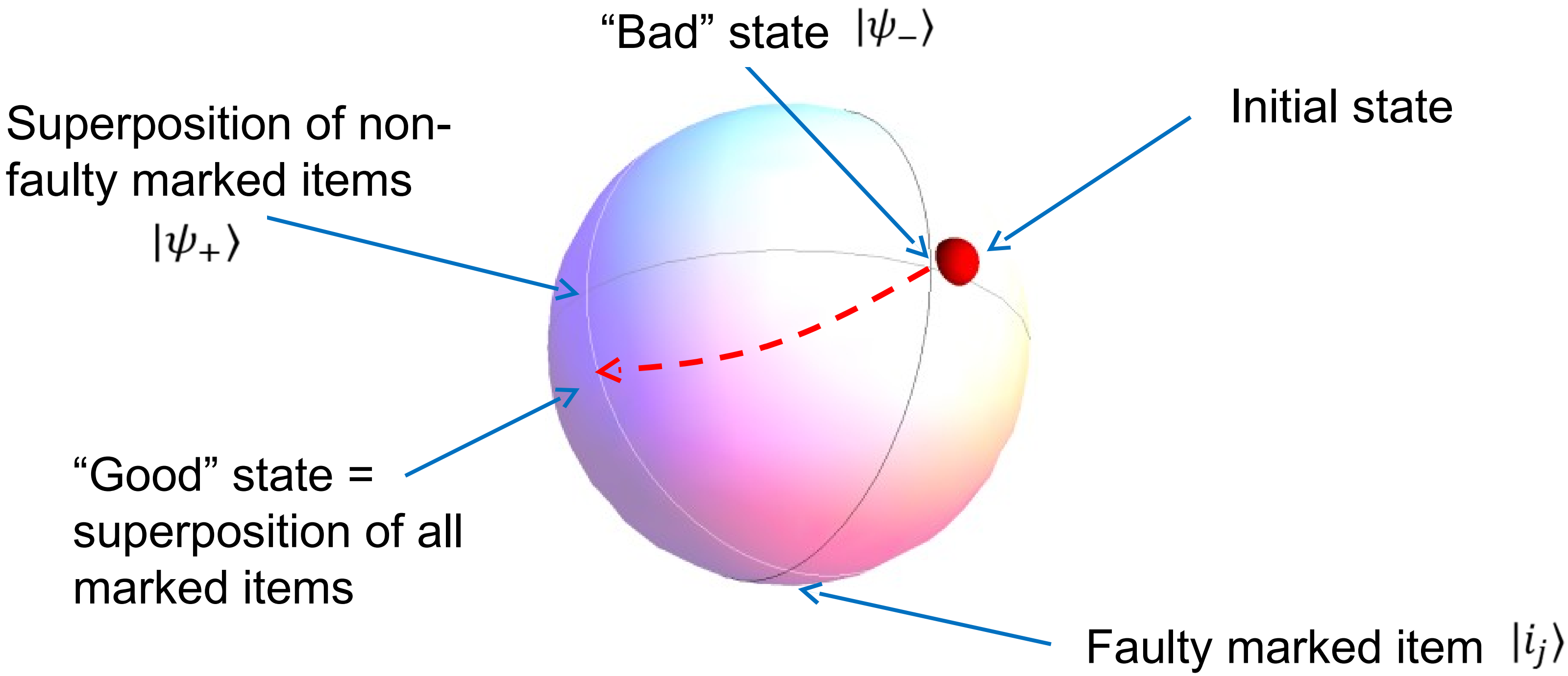
First peak: analysis

- Grover's search without errors is a rotation in two dimensions



First peak: analysis

- Grover's search with one faulty and k-1 non-faulty marked elements is a rotation in three dimensions



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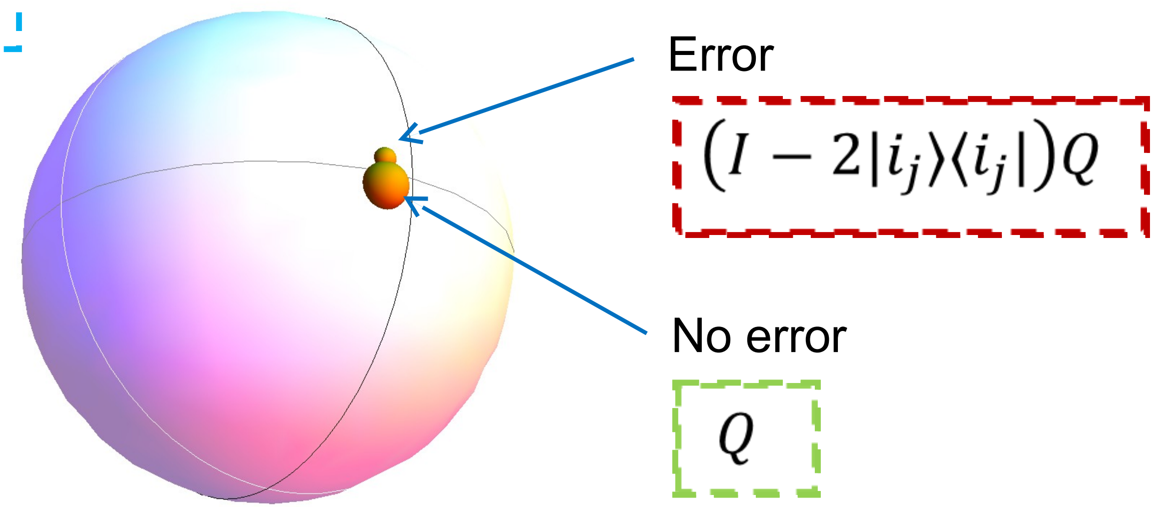
Volume of the ball = probability



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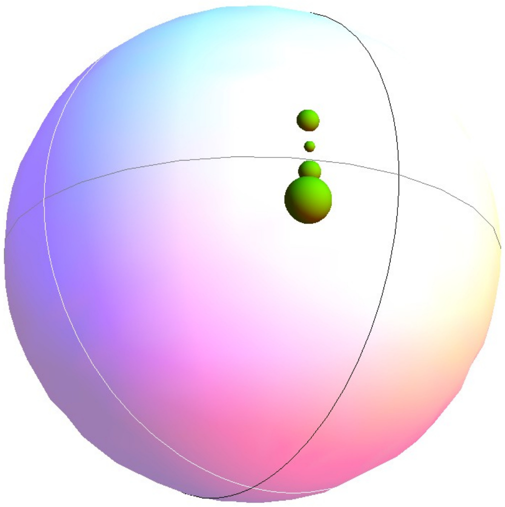

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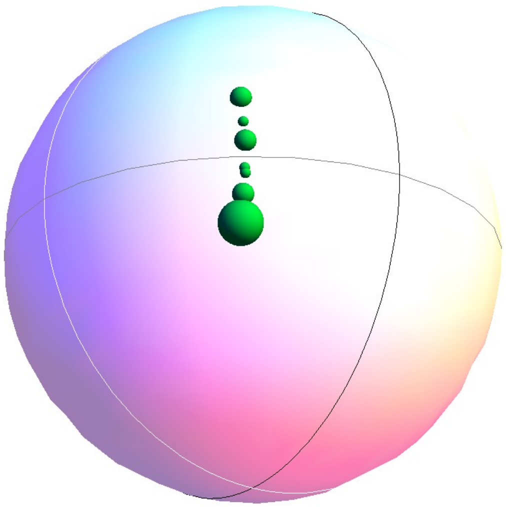

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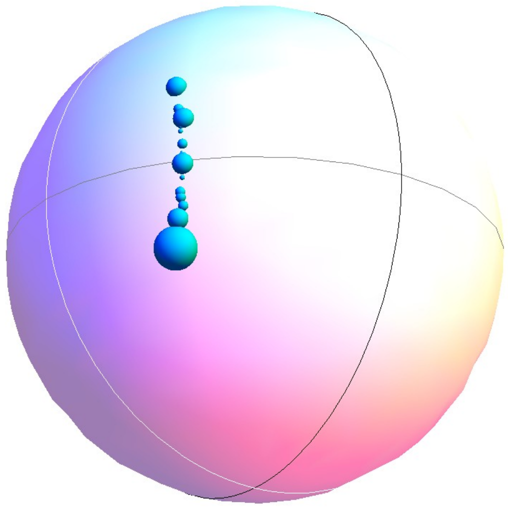

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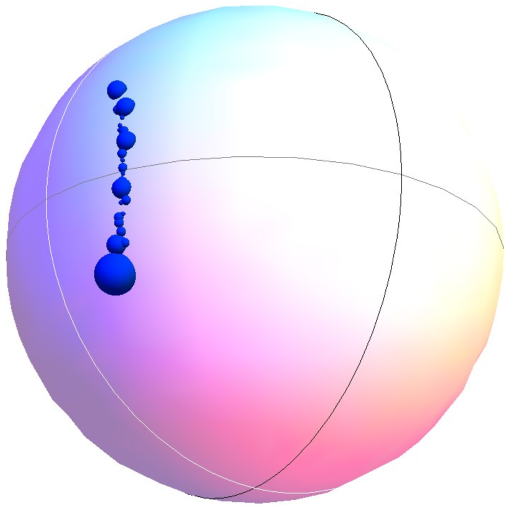

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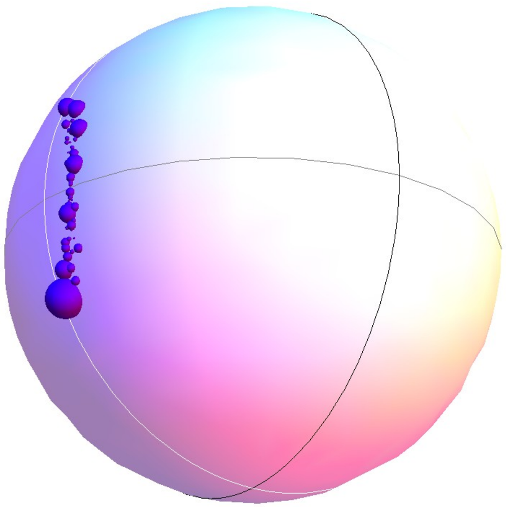

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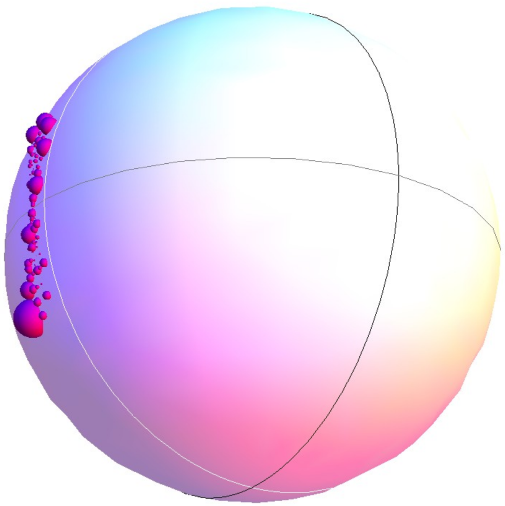

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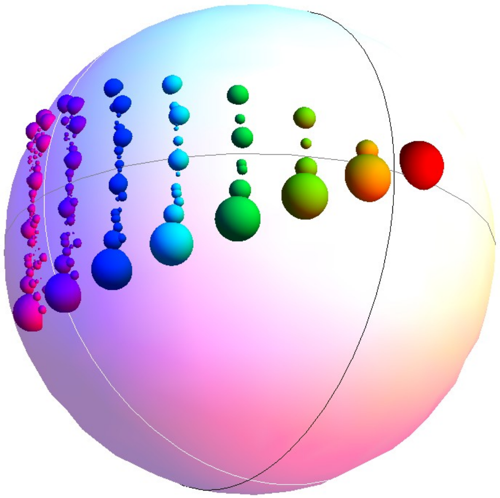

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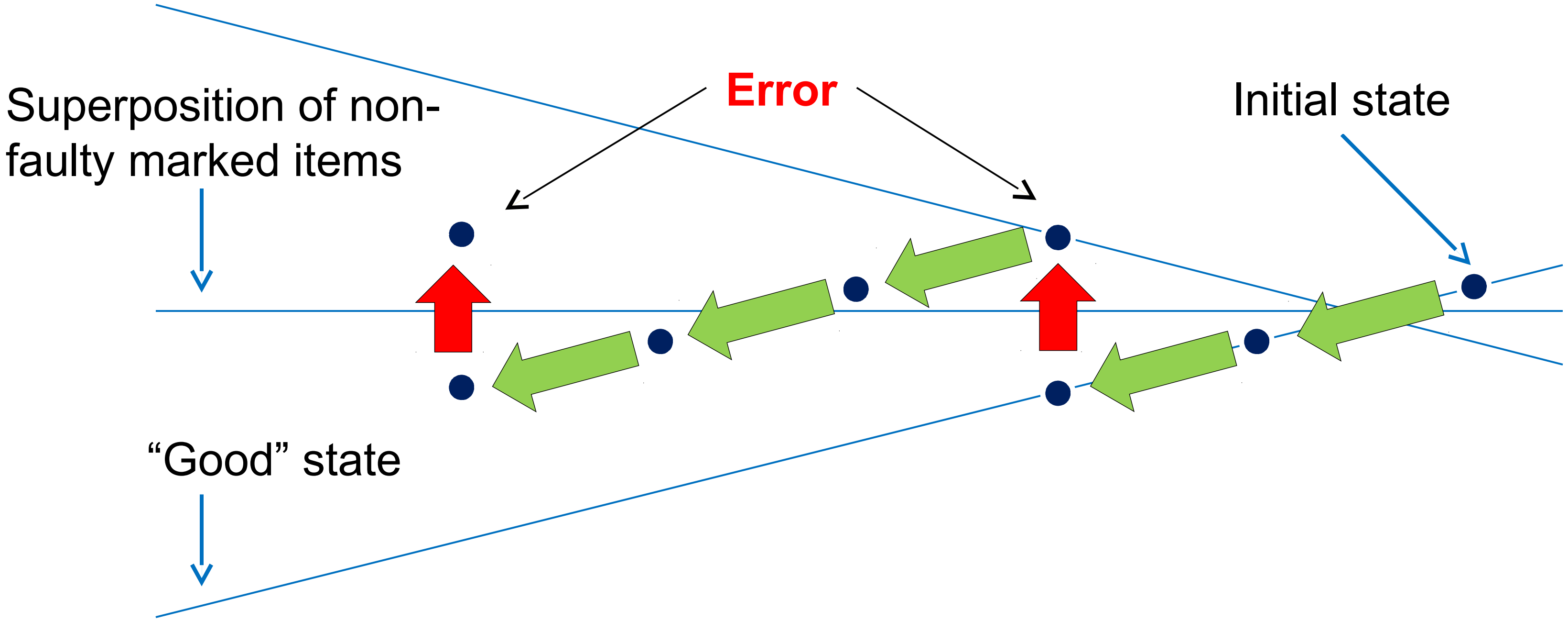

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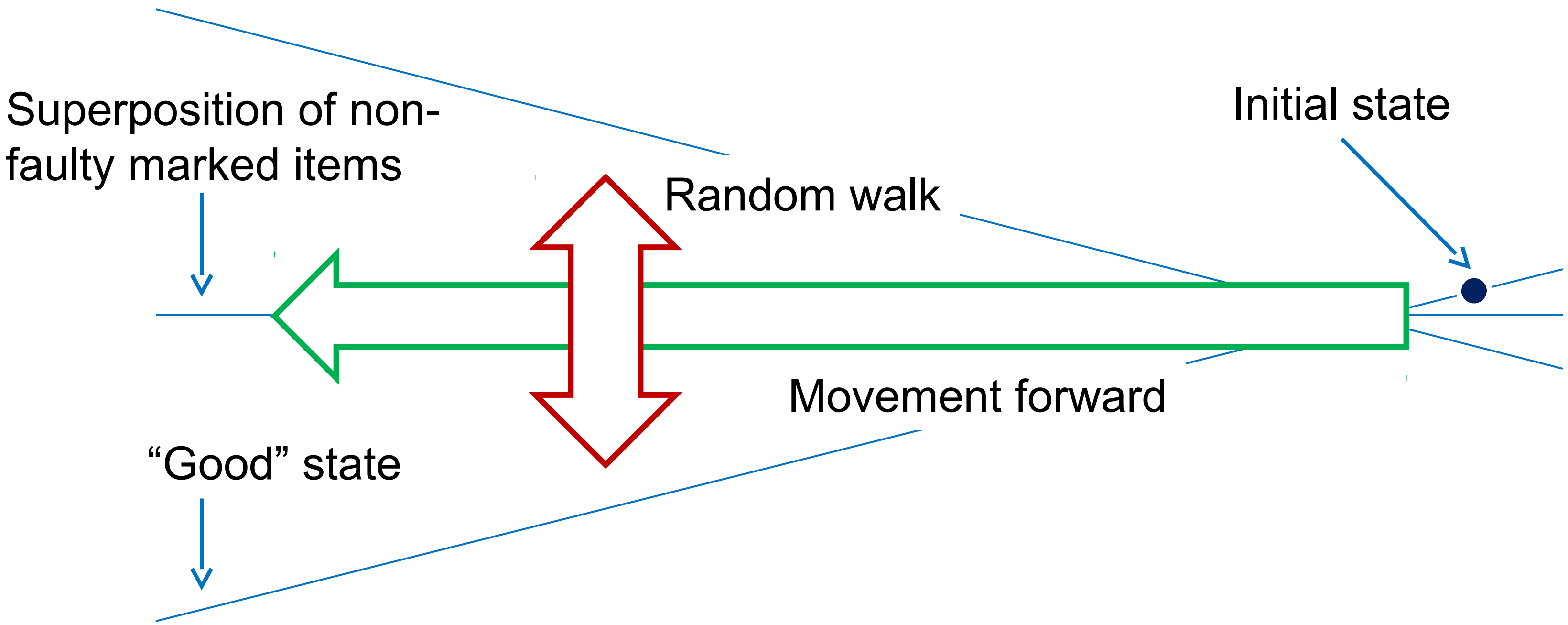

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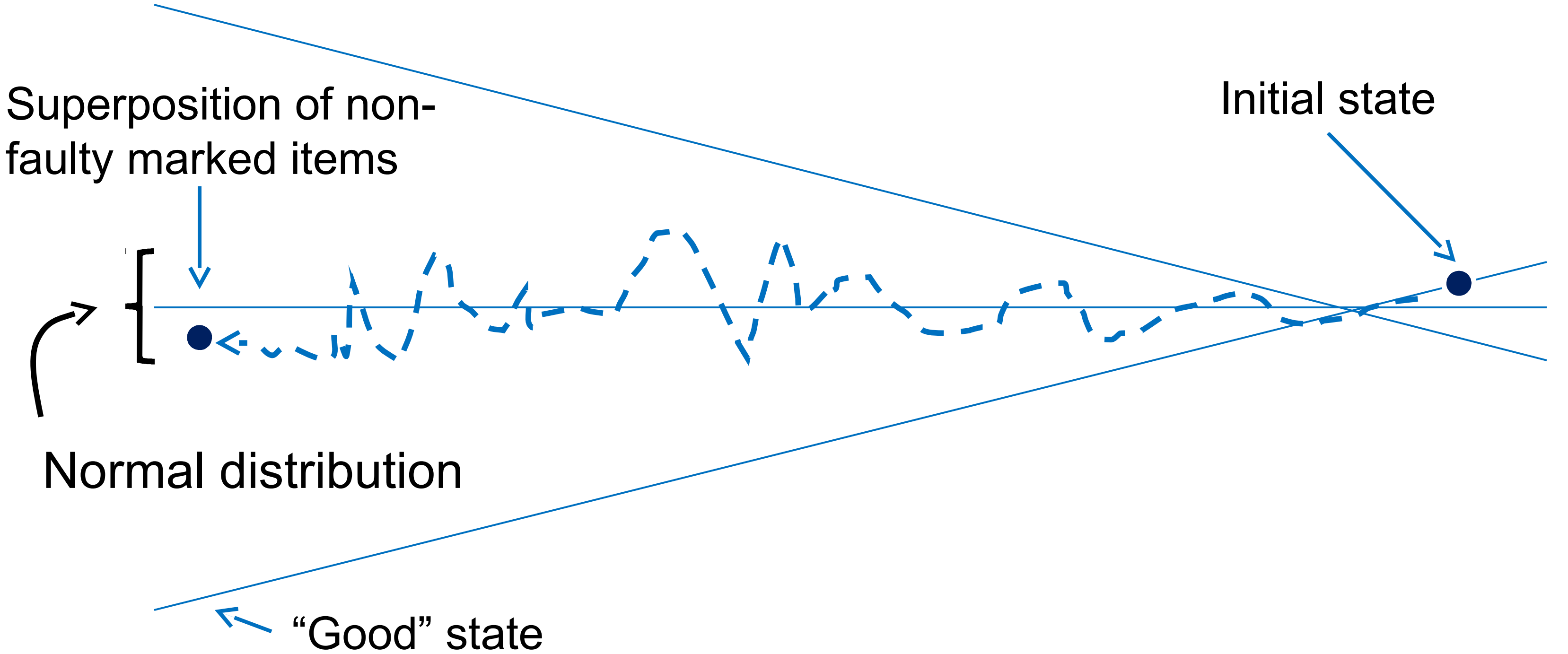

Limiting state (once again)

- The limiting state

$$\rho_{\lim} = \frac{1}{3}|i_j\rangle\langle i_j| + \frac{1}{3}|\psi_+\rangle\langle\psi_+| + \frac{1}{3}|\psi_-\rangle\langle\psi_-|$$

is just a uniformly covered sphere.

Open questions

- Can we prove a bound on a number of queries for any quantum algorithm (similarly to [RS08]) ?

- Similar analysis for other quantum search algorithms.

Thank you !

Algorithmic application: fuzzy search

- We are given a pattern P and a set of strings { S1, …, SN }

Pattern

P :	1	1	…	0	…	1	0	1	0

Search space

S1 :	1	0	…	0	…	0	1	1	0
S2 :	0	1	…	1	…	1	0	1	1
…	…	…	…	…	…	…	…	…	…
SN :	0	1	…	1	…	0	1	1	0

On step t query returns if P[t] = Sj[t]

- Find a string Sj which is approximately equal to P.