INEQUALITIES FOR THE RANKS OF QUANTUM STATES

CEQIP 2014

AND THE STRUCTURE OF ENTANGLEMENT IN MULTI-DIMENSIONAL SYSTEMS

Based on work done in collaboration with:

Josh Cadney, Noah Linden and Andreas Winter

(and related work with Marti Perarnau-Llobet and Julio I. de Vicente)

Motivation

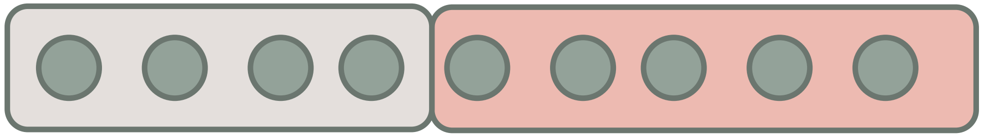
- Marginal entropies determine the distribution of information in multipartite quantum systems

- => we want to understand the structure of complex multipartite quantum systems

Outline

1. Introduction: Entropy inequalities
2. Inequalities for the rank
3. Applications in entanglement classification
4. Conclusion & a hypothesis

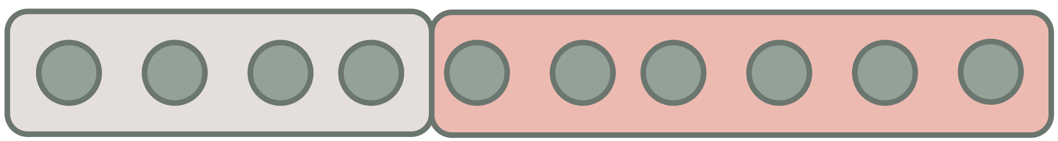
Multipartite entropy distributions

$\varrho =$ 

A $\quad\quad$ $\bar{A}$

$2^n - 2$ possible subsystems ρ_A with different entropy

$\equiv$

$|\psi\rangle =$ 

A $\quad\quad$ $\bar{A}$

$2^{n-1} - 1$ possible subsystems ρ_A with different entropy

(encodes entanglement distribution)

α

$\alpha = 0$: $S_0(\varrho) = \log rank(\varrho)$???

$\alpha \in (0,1)$: No inequality possible [3]

$\alpha = 1$: Rich structure (SSA, WM, etc.) [1,2]

Renyi-entropies $S_\alpha(\varrho) \coloneqq \frac{1}{1-\alpha} \log Tr(\varrho^\alpha)$

$\alpha \in (1, \infty)$: No linear inequalities possible [3]

SSA: $S(\varrho_A) + S(\varrho_{ABC}) \leq S(\varrho_{AB}) + S(\varrho_{AC})$
WM: $S(\varrho_A) + S(\varrho_B) \leq S(\varrho_{AC}) + S(\varrho_{BC})$

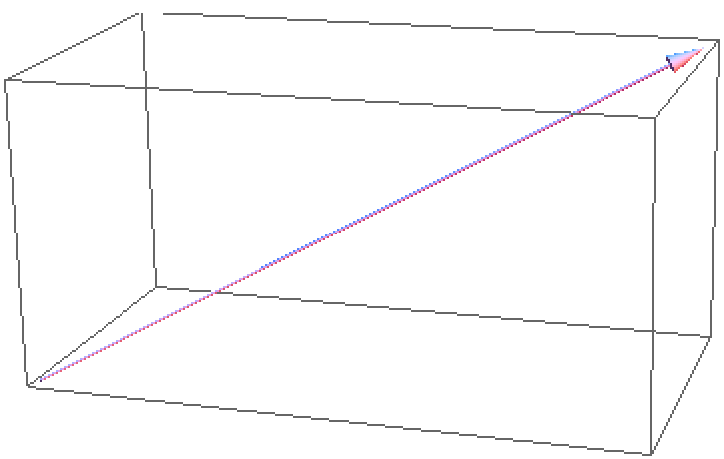
∞

[1] H. Araki and E.H. Lieb, *Entropy Inequalities*, Comm. Math. Phys., **18**, 160–170 (1970).
[2] E.H. Lieb and M.B. Ruskai, *Proof of the Strong Subadditivity of Quantum-Mechanical Entropy*, J. Math. Phys., **14**, 1938–1941 (1973).
[3] N. Linden, M. Mosonyi and A. Winter, *The structure of Renyi entropic inequalities*, Proc. R. Soc. A **469**(2158):20120737, (2013)

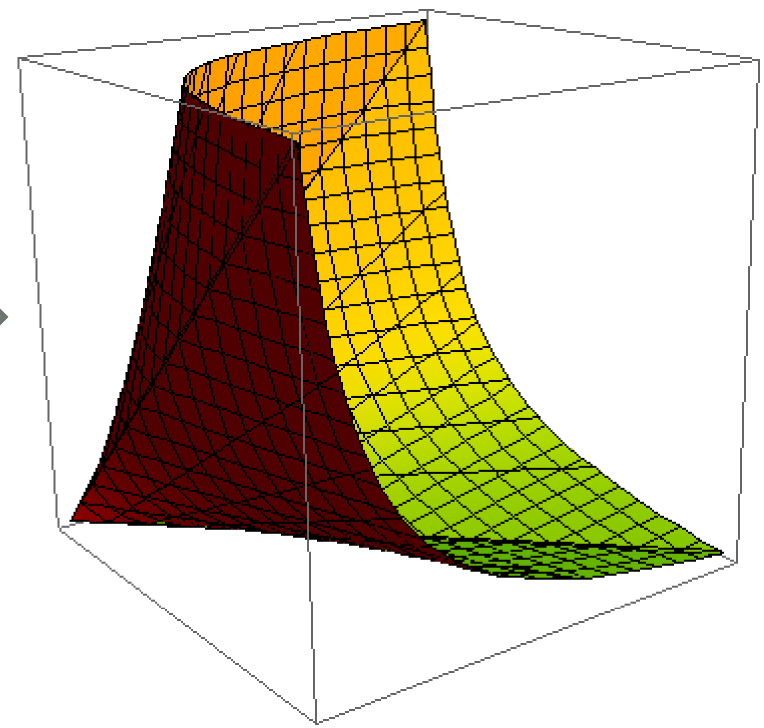
Rank vectors

- $r_j := rank(\rho_j)$

Given $\vec{r} \in \mathbb{N}^m$ and $1 \leq m \leq 2^{n-1} - 1$ ⇨ What ranks are actually possible?



?



MH and J.I. de Vicente, *The structure of multidimensional entanglement in multipartite systems*, Phys. Rev. Lett. **110**, 030501 (2013)

Tripartite states: ϱ_{ABC}

$$S_\alpha(\varrho) \leq S_\beta(\varrho) \text{ for } \alpha \geq \beta$$

$$S(\varrho_A) + S(\varrho_{ABC}) \leq S(\varrho_{AB}) + S(\varrho_{AC}) \leq S_0(\varrho_{AB}) + S_0(\varrho_{AC})$$

apply filtering

$$S_0(\varrho_A) \leq S_0(\varrho_{AB}) + S_0(\varrho_{AC})$$

$$r_A \leq r_{AB} r_{AC}$$

$$|\Psi_{ABC}\rangle = \sum_{i=1}^{r_A} |i\rangle \otimes |\Phi_i\rangle \qquad r_{AB} = r_{AC} \leq {r_A}^2$$

J. Cadney, MH, N. Linden and A. Winter, *Inequalities for the ranks of quantum states*, Linear Algebra and Applications, **452**, pp. 153-171 (2014)

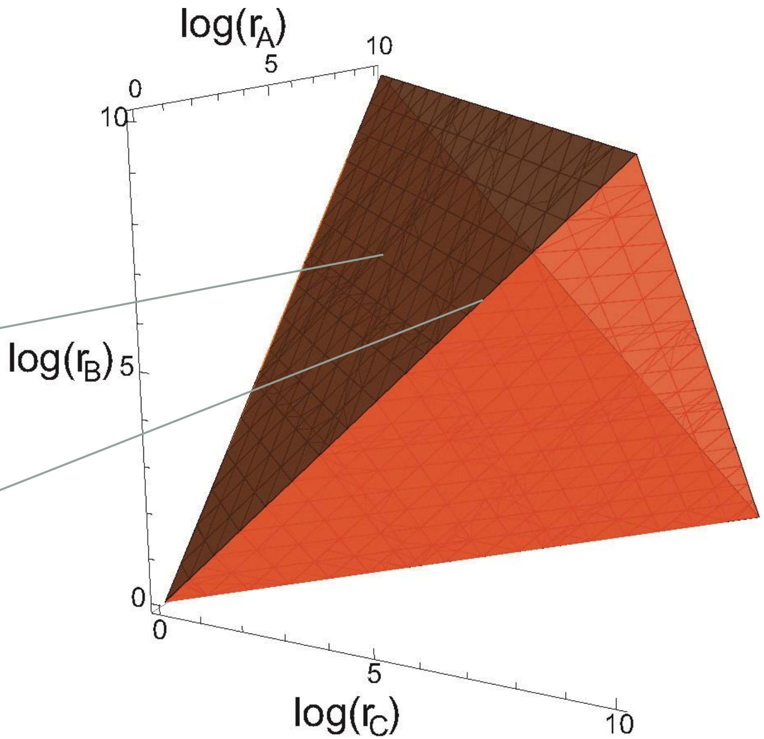
Convex cones

- $r_j(|\psi\rangle) \coloneqq \log(rank(\rho_j))$
- $\vec{r}(|\psi\rangle \otimes |\phi\rangle) = \vec{r}(|\psi\rangle) + \vec{r}(|\phi\rangle)$

$r_B r_C \geq r_A$

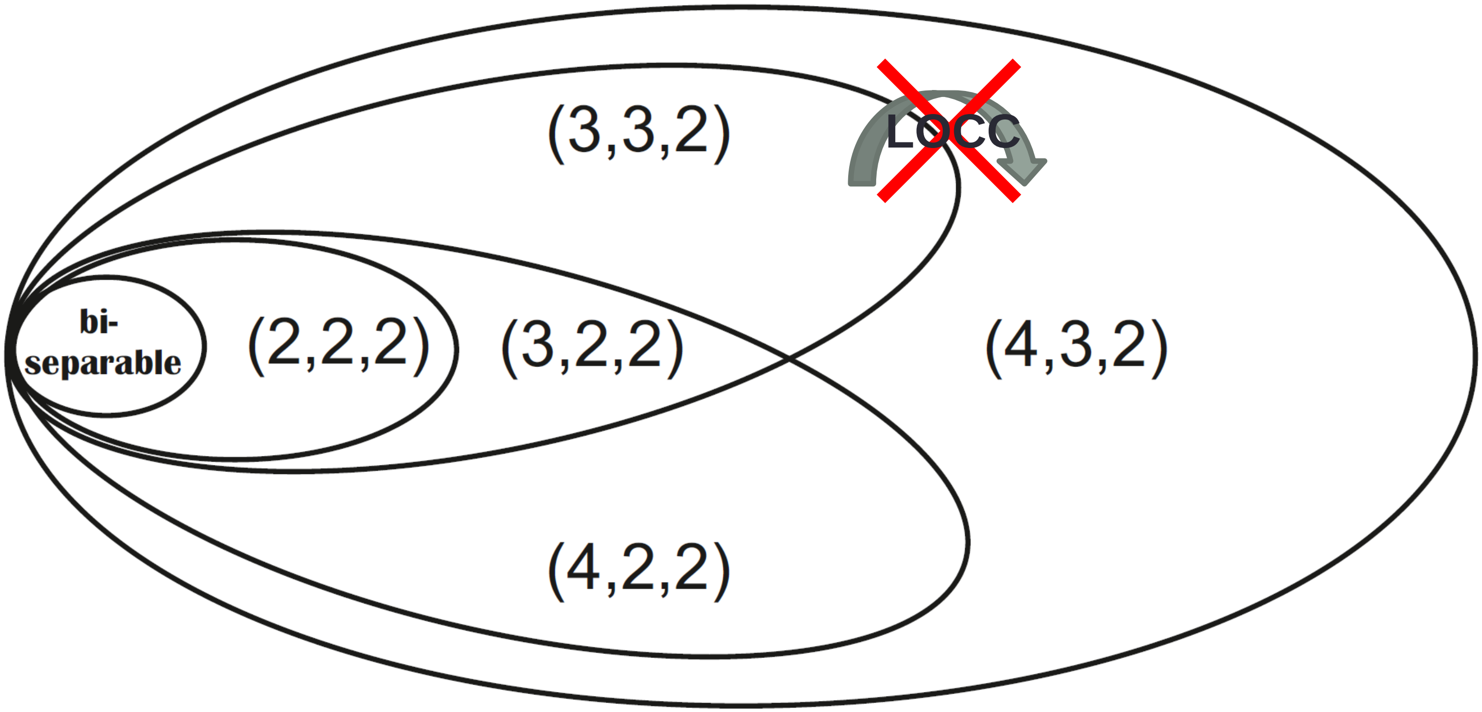
Extremal ray:

$$|\Psi_{ABC}\rangle = \sum_{i=1}^{r_A} |i\rangle \otimes |\Phi_i\rangle$$



Applications in entanglement classification

- $r_j := rank(\rho_j)$ and $r_i \leq r_{i+1}$
- $d_\mathrm{j} := \min_{D(\rho)} \max_{\psi_i} r_j{}^i$



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Conclusion & Open Problems

- New inequalities for the ranks and new techniques for finding them

- Open Problems: rank vectors & inequalities for $\mathrm{n} \geq 4$?

From s.s.a. and w.m. we know: (1) $r_A r_B \geq r_{AB}$

(2) $r_{AB} r_{AC} \geq r_A$ ← elementary proof (applies to (1) and (2))

(3) $r_{AB} r_{AC} r_{BC} \geq {r_A}^2$

Hypothesis: (4) $r_{BC} \leq r_{AC}\, r_{AB}$?

We also have:(asymptotic) construction for all states satisfying (1-4)

J. Cadney, MH, N. Linden and A. Winter, *Inequalities for the ranks of quantum states*, Linear Algebra and Applications, **452**, pp. 153-171 (2014)

Thank you!

- Rank inequalities

J. Cadney, MH, N. Linden and A. Winter, *Inequalities for the ranks of quantum states*, Linear Algebra and Applications, **452**, pp. 153-171 (2014)

- Measures of GME-dimensionality and Witness constructions

MH and J.I. de Vicente, *The structure of multidimensional entanglement in multipartite systems*, Phys. Rev. Lett. **110**, 030501 (2013)

MH M. Perarnau-Llobet and J.I. de Vicente, *The entropy vector formalism and the structure of multidimensional entanglement in multipartite systems*, Phys. Rev. A **88**, 042328 (2013)

The canonical approach

$\hat{W} := \alpha\mathbb{I} - |\psi\rangle\langle\psi|$ with $\alpha = \max_{\varphi\in\mathcal{S}} |\langle\psi|\varphi\rangle|^2$

e.g. for $\mathcal{S} \equiv (2,3,4)$? ⟹ hard

Detecting (multipartite) entanglement

$$S_L(\rho) = 2(1 - Tr(\rho^2)) \qquad \vec{S_L}(\rho) \geq \vec{W}(\rho)$$

$$\sqrt{[\vec{W}(\rho)]_i} = \sum_{\{\alpha,\beta \epsilon C\}} \left(|\langle\alpha|\varrho|\beta\rangle| - \min_{\{A_k\}} \left(\sum_{k=1}^{|\{A_k\}|} \sqrt{\langle\alpha\beta|P_{A_k\to A_{k'}}\varrho^{\otimes 2}P_{A_k\to A_{k'}}|\alpha\beta\rangle} \right)\right) \; *$$

For X-matrices:
$$[\vec{W}(\rho)]_1 \leq \max_i \left(\inf_{\{p_k,|\psi_k\rangle\}} \sum_k p_k \, [\vec{S}(|\psi_k\rangle)]_i \right) \leq [\vec{W}(\rho)]_1$$

$$[\vec{W}(\rho)]_{2^{n-1}-1} \leq \inf_{\{p_k,|\psi_k\rangle\}} \sum_k p_k \min_i \left([\vec{S}(|\psi_k\rangle)]_i \right) \leq [\vec{W}(\rho)]_{2^{n-1}-1} \; **$$

* *MH and J. I. de Vicente*, Phys. Rev. Lett. **110**, 030501 (2013)
** *S. Rafsanjani, MH, C. Broadbent and J.H. Eberly*, Phys. Rev. A. **86**, 062303 (2012)

Measures and witnesses

Convex roof measures of multipartite entanglement:

$$\overrightarrow{S_\alpha}(\rho) := \inf_{\{p_k, |\psi_k\rangle\}} \sum_k p_k \overrightarrow{S_\alpha}(|\psi_k\rangle)$$

If $\overrightarrow{S_\alpha}(|\psi_k\rangle)$ is the full (and ordered) entropy vector:

$[\overrightarrow{S_\alpha}(\rho)]_i > 0 \Leftrightarrow \rho$ not $\mathrm{n} - \log(i+1) + 1$-separable,

e.g. last entry greater 0 implies genuine multipartite entanglement

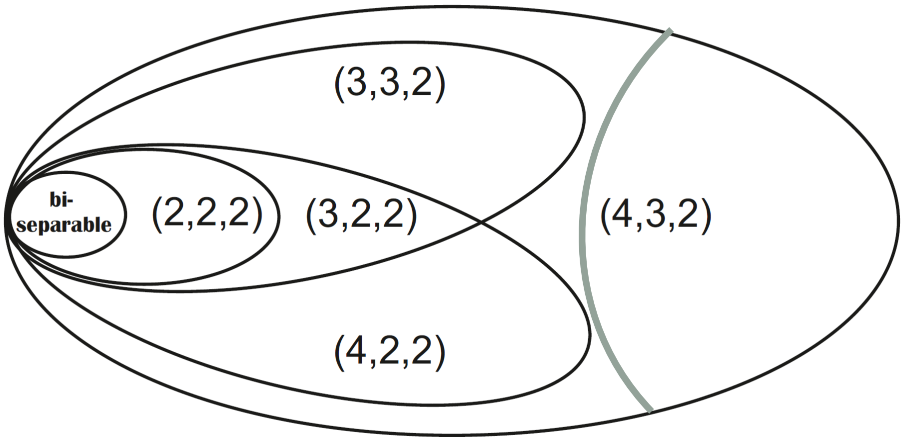
Connecting to dimensionality (i.e. ranks)

- $r_j := rank(\rho_j)$ and $r_i \leq r_{i+1}$
- $d_j \; := \min_{D(\rho)} \max_{\psi_i} r_j{}^i$

Since $S^j{}_2 \geq -\log_2(1 - \frac{W_j}{2})$

⇩

$$d_\mathrm{j} \geq \frac{1}{1 - \frac{W_j}{2}}$$



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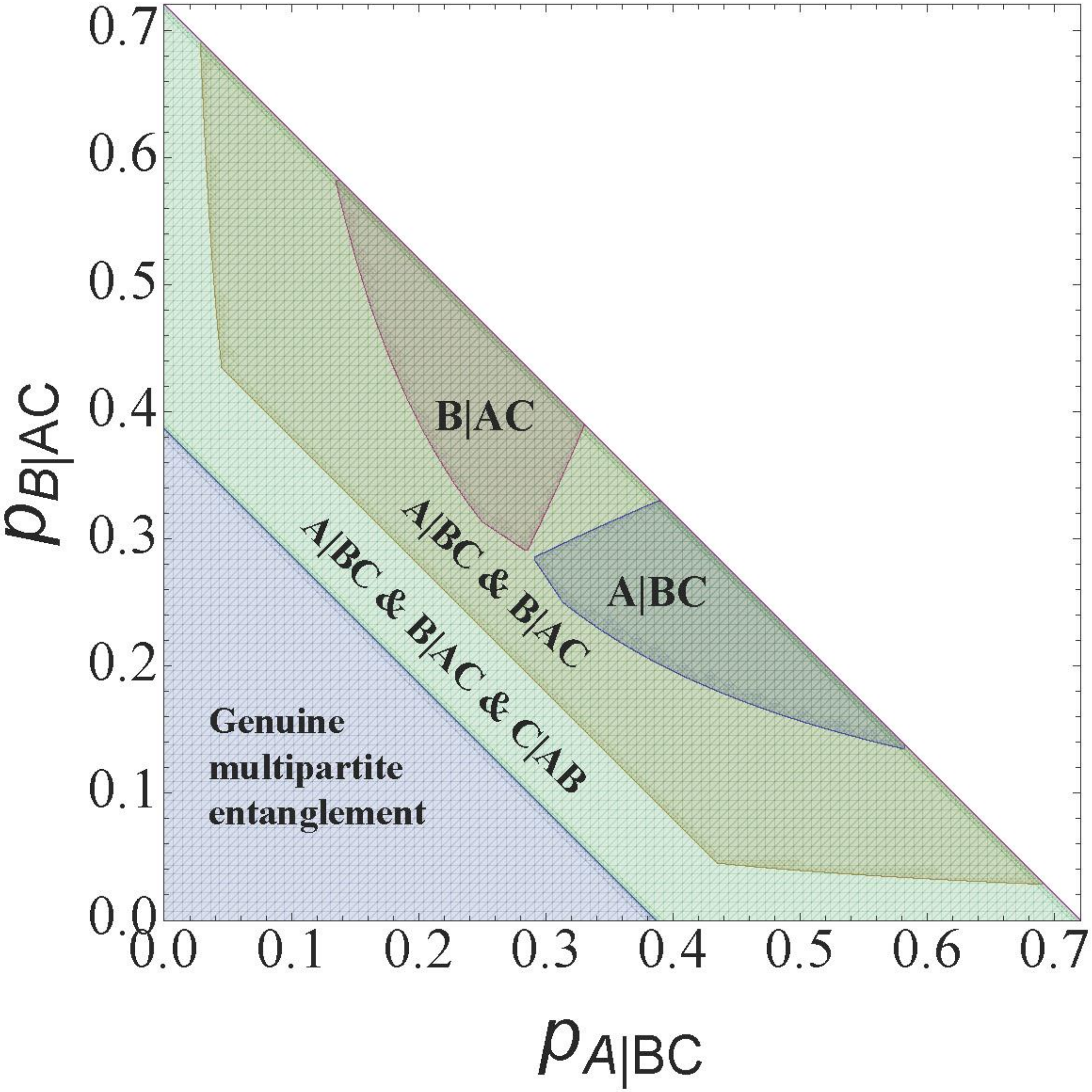
- Some examples

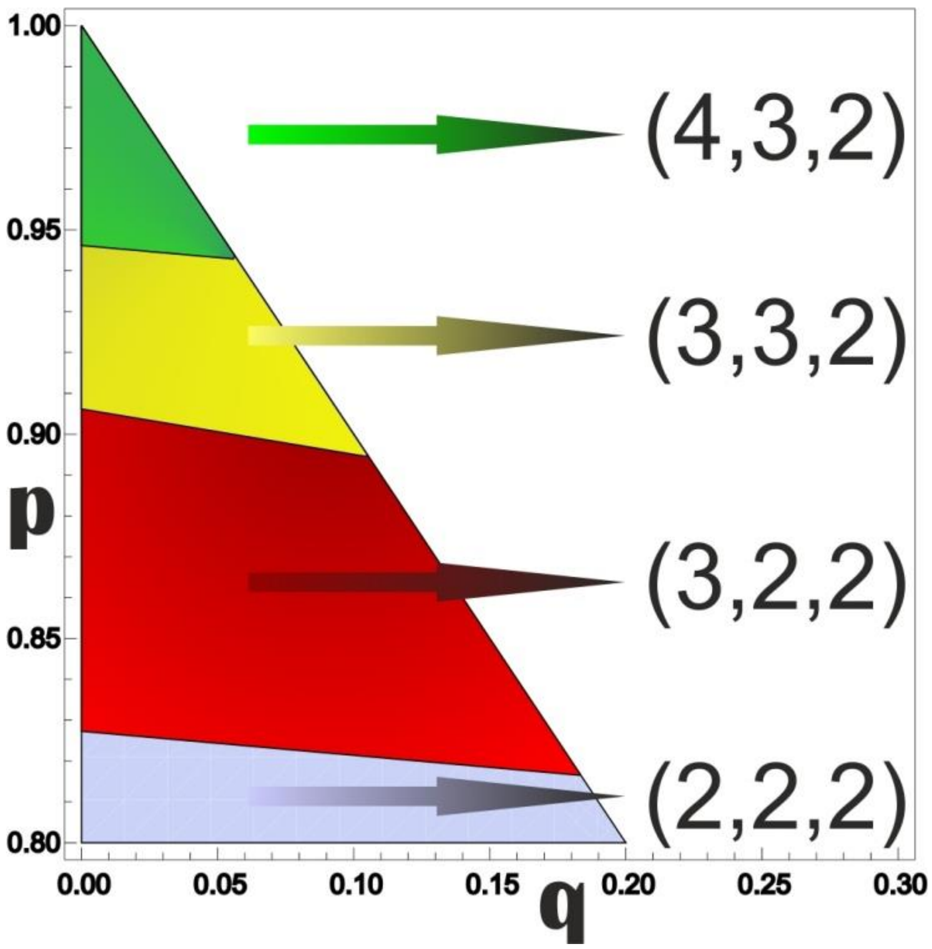
$$\varrho = p_{A|BC}\frac{\mathbb{I}_A}{2}\otimes\sigma_{BC} + p_{B|AC}\frac{\mathbb{I}_B}{2}\otimes\sigma_{AC} + p_{C|AB}\frac{\mathbb{I}_C}{2}\otimes\sigma_{AB} + (1 - p_{A|BC} - p_{C|AB} - p_{B|AC})\sigma_{ABC}$$

$$p_{C|AB} = 0.28$$



$$|\psi_{(432)}\rangle = \frac{1}{2}(|000\rangle + |111\rangle + |012\rangle + |123\rangle)$$

$$\varrho = p|\psi_{(432)}\rangle\langle\psi_{(432)}| + q\sigma_{dp} + (1-q-p)\frac{1}{64}\mathbb{I}$$



Improving detection for known density matrices: the normal form

$$|\psi\rangle = \sqrt{1-2\varepsilon}|000\rangle + \sqrt{\varepsilon}|111\rangle + \sqrt{\varepsilon}|222\rangle$$

$$\begin{pmatrix} \frac{1}{\sqrt{3(1-2\varepsilon)}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3\varepsilon}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3\varepsilon}} \end{pmatrix}^{\otimes 3}$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$$