(Qubit) Stabilizer States are Complex Projective 3-Designs

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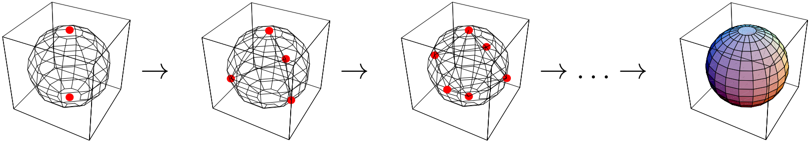

Outline

- Motivation
- Complex Projective t-Designs
- Stabilizer States
- Combining the two concepts $\Rightarrow$ our Result
- Application: Phase Retrieval using Stabilizer States

Motivating Problem

Consider N mutually opposing küngs on a sphere S^{D-1}. Distribute them as far away from each other as possible.

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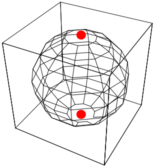
$N = 2$:  vs.  $\Rightarrow$ 

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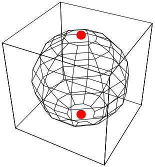
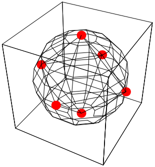
$N = 2$:  vs.  $\Rightarrow$ 

$N = 6$:  $\Rightarrow$ 

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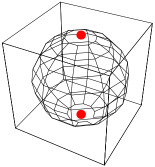
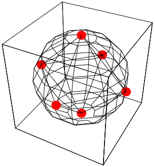
$N = 2$:  vs.  $\Rightarrow$ 

$N = 6$:  $\Rightarrow$ 

Conceptually related to the Sphere Packing problem

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- Frame Potential: $\mathcal{F}_t(x_1, \ldots, x_N) = \frac{1}{N^2} \sum_{i,j=1}^{N} |\langle x_i, x_j \rangle|^{2t}$
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- Fact: For each $t = 1, 2, \ldots$ the ground state energy of $\mathcal{F}_t$ is known analytically: Welch Bound / Sidelnikov Inequality

Complex Projective t-Designs

Consider a set $X = \{x_1, \ldots, x_N\} \subset \mathbb{C}^D$ of unit vectors.

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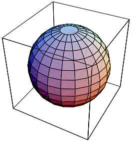
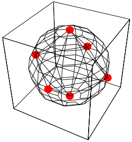


$$\int_{\mathbb{C}P^{D-1}} p(x)\mathrm{d}x = \frac{1}{n}\sum_{i=1}^{n} p(x_i)$$



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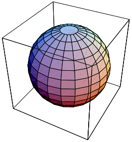

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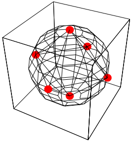


  for any (hermitian) polynomial $p : \mathbb{C}P^{D-1} \to \mathbb{R}$ of degree at most t.
- Definition (Representation Theory): X is a t-Design iff

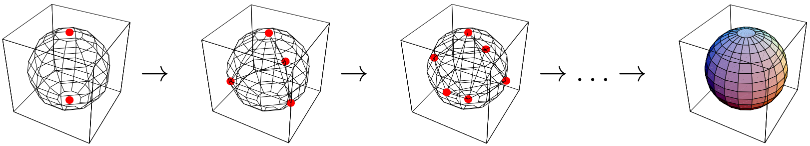
$$\sum_{i=1}^{N} (|x_i\rangle\langle x_i|)^{\otimes t} \propto P_{\mathrm{Sym}^t(\mathbb{C}^d)}.$$

Complex Projective t-Designs

Designs are known to exist for every t and D.

- Hierarchy: every t-design is a $(t-1)$-design, etc.
- Tight frames and ONB's are 1-designs.
- SIC-POVM's and MUB's are 2 designs.
- Higher order designs notoriously difficult to find.
- Approximate t-designs can be constructed efficiently.

Operational interpretation: Sampling iid from a t-design reproduces the Haar measure up to t-th moments.



Stabilizer States

Construction:

- 'Paulis': $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$.
- Pauli Group: $G_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$
- G_n constists of all possible n-fold tensor products of elements of G_1.
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- Rich combinatorial structure governed by discrete symplectic geometry.
- Many applications in Quantum Info, e.g. error correcting codes, Gottesman-Knill Theorem.

Our Result

Theorem [RK, David Gross]:

- n-Qubit Stabilizer States are Complex Projective 3-designs ($n \geq 1$ arbitrary).
- The same is NOT true for qudits ($d \neq 2$).

Corollary: (Weighted) 3-designs for arbitrary dimensions can be obtained by projecting down higher dimensional n-qubit stabilizer states.

Proof Sketch

Idea: show that stabilizer states minimize the frame Potential

$$\mathcal{F}_3(X) = \frac{1}{N^2} \sum_{i,j=1}^{N} |\langle x_i, x_j \rangle|^6$$

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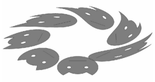
- Overlaps $|\langle x_i, x_j \rangle|^2$ can be calculated using discrete symplectic geometry. 

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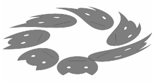
- Overlaps $|\langle x_i, x_j \rangle|^2$ can be calculated using discrete symplectic geometry. 

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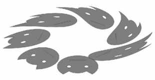
- Overlaps $|\langle x_i, x_j \rangle|^2$ can be calculated using discrete symplectic geometry. 

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$\Rightarrow$ Counting Problem:
$\mathcal{F}_3(X)$ equals the known minimum (Welch bound) if and only if $D = 2^n$ (qubit stabilizer states).

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$\Rightarrow$ Example: Phase Retrieval with stabilizer states

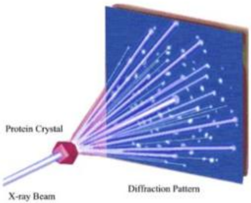
Phase Retrieval / Pure State Tomography

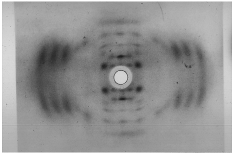
Recover an unknown vector $x \in \mathbb{C}^D$ from quadratic measurements

$$y_i = |\langle a_i, x\rangle|^2 = \mathrm{tr}\left(|a_i\rangle\langle a_i||x\rangle\langle x|\right) \quad i = 1, \ldots, m.$$

How many measurements are sufficient?

- Classical Motivation: from diffraction imaging:





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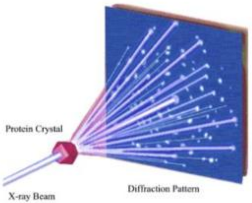
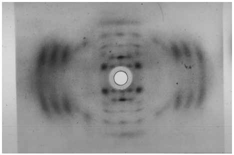




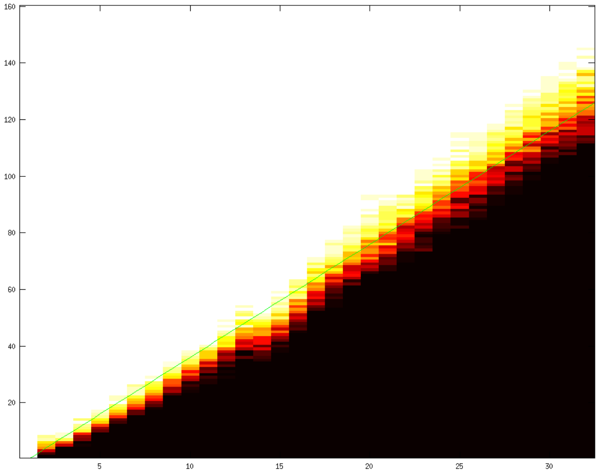
- Quantum Motivation: Pure State Tomography with rank-one measurements.

Phase Retrieval: Brainstorming

- Task: Recover a state $|x\rangle\langle x|$ in D^2-dimensional Hilbert space.
- Idea: (from compressed sensing): use random measurements and additional structure.
- Random measurements $\Rightarrow$ random overlaps with stabilizer states (3-design instead of Haar random).
- additional structure: low rank (one!) $\Rightarrow$ trace-norm minimization as a good proxy

$$\begin{aligned} X_{\text{guess}} &= \operatorname{argmin} \|Z\|_1 \\ \text{subject to} \quad & \operatorname{tr}(Z|a_i\rangle\langle a_i|) = \operatorname{tr}(|x\rangle\langle x||a_i\rangle\langle a_i|) \\ & Z \geq 0. \end{aligned}$$

Phase Retrival: Numerical Experiments



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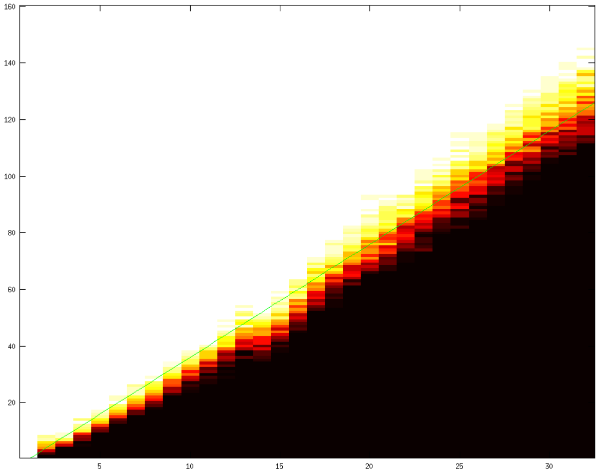

- ☺ works way better than anticipated: $m =\simeq 4d - 4$ sufficient instead of $m = \mathcal{O}(d^{5/3} \log^2 d)$
[D. Gross, F. Krahmer, RK, 2013]

Summary

> *n*-Qubit Stabilizer States are complex projective 3 designs ($n \geq 1$ arbitrary).

- ☺ Explicit 3-design with nice structure.
- ☹ not a tight 3-design (there are superpolynomially many stabilizer states)
- ☺ Weighted 3-designs in arbitrary dimensions can be obtained by projecting down larger Qubit stabilizer states.
- ☺ Promising applications in Phase-Retrieval / low rank matrix recovery (ongoing work)