Additivity & quantum channel capacity: an old problem revisited

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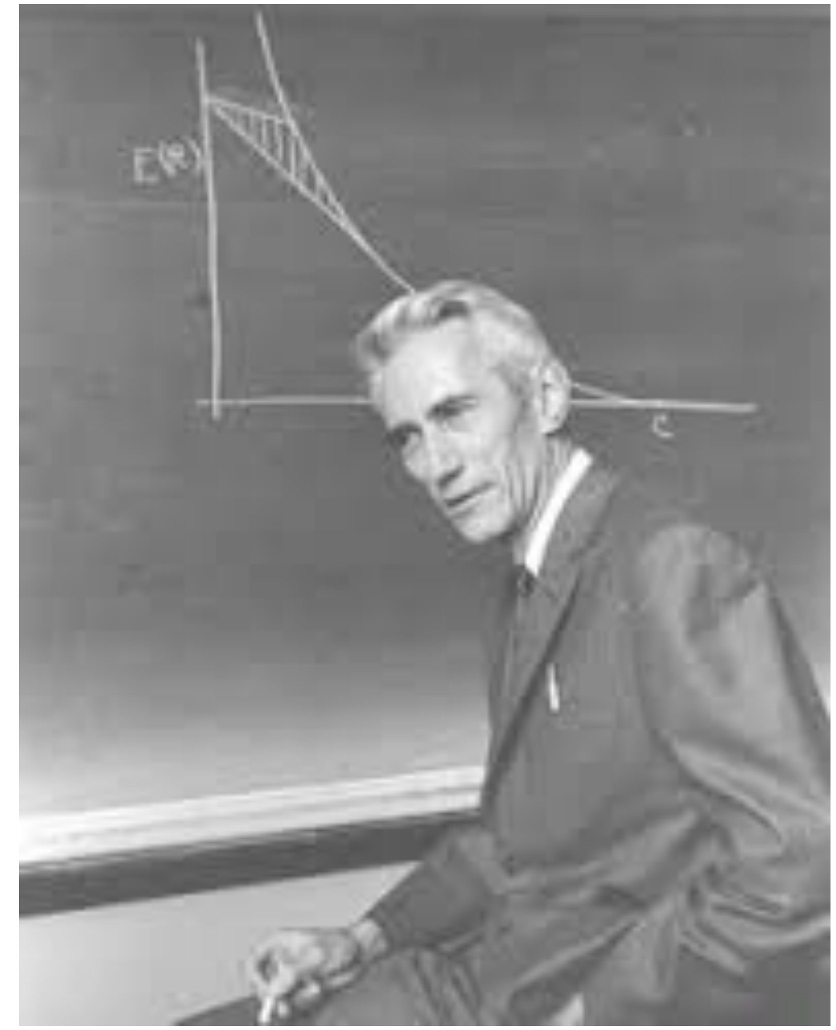
Motivation

Shannon's seminal 1948 article

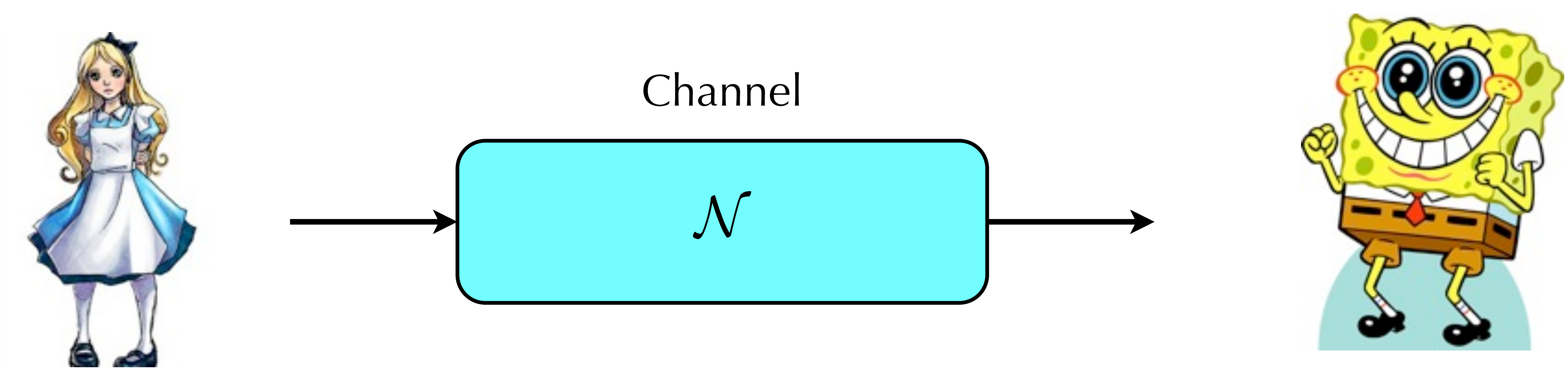
Entropy - uncertainty
Source coding
Channel coding theorem

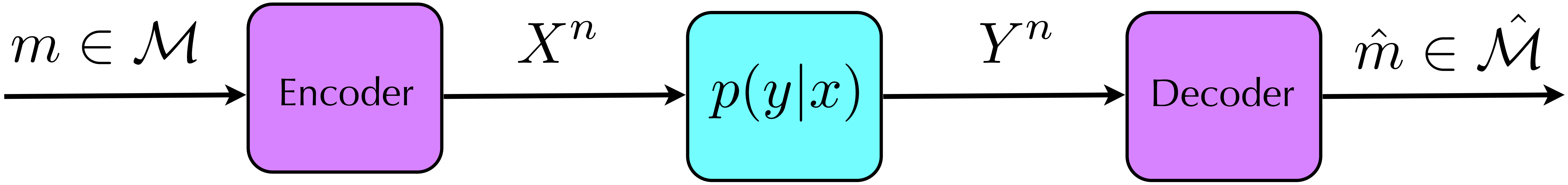
Ultimate capability for physical systems to preserve information & correlations

Claude E. Shannon



Father of Information Theory





X - random variable taking values $x \in \mathcal{X}$

with probability $P_X(x)$

Error: $Pr[m \neq \hat{m}] \leq \epsilon$

Defines an (n, ϵ) - code

Channel coding theorem

Using n copies of the channel can code at rate

$$R = \frac{1}{n} \log |\mathcal{M}|$$

with an exponentially small probability of error if

$$R < C$$

the probability of decoding the information correctly goes to zero if

$$R > C$$

Shannon's noisy channel coding theorem

The capacity of a noisy classical channel $\mathcal{N}$ is the maximum of the mutual information obtained over all possible input distributions $P_X(x)$ for the random variable X

$$\mathcal{C}(\mathcal{N}) = \max_{P_X(x)} I(X:Y).$$

mutual information $I(X:Y)$ is given by

$$I(X:Y) = H(X) + H(Y) - H(X,Y)$$

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Single- letter

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We want this in the quantum setting

Types of information

Various types of information can be sent over a quantum channel

- classical information
- private information
- quantum information

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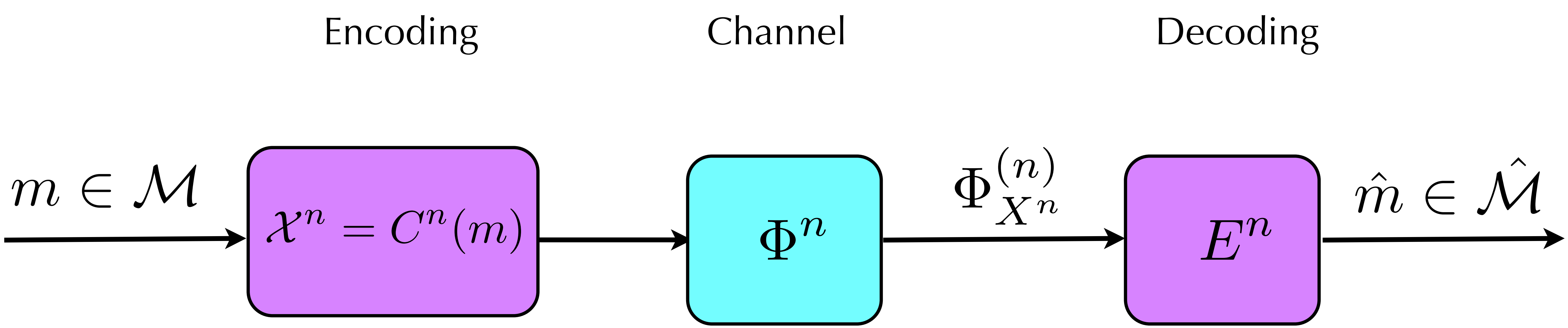
Transmission over a classical-quantum channel

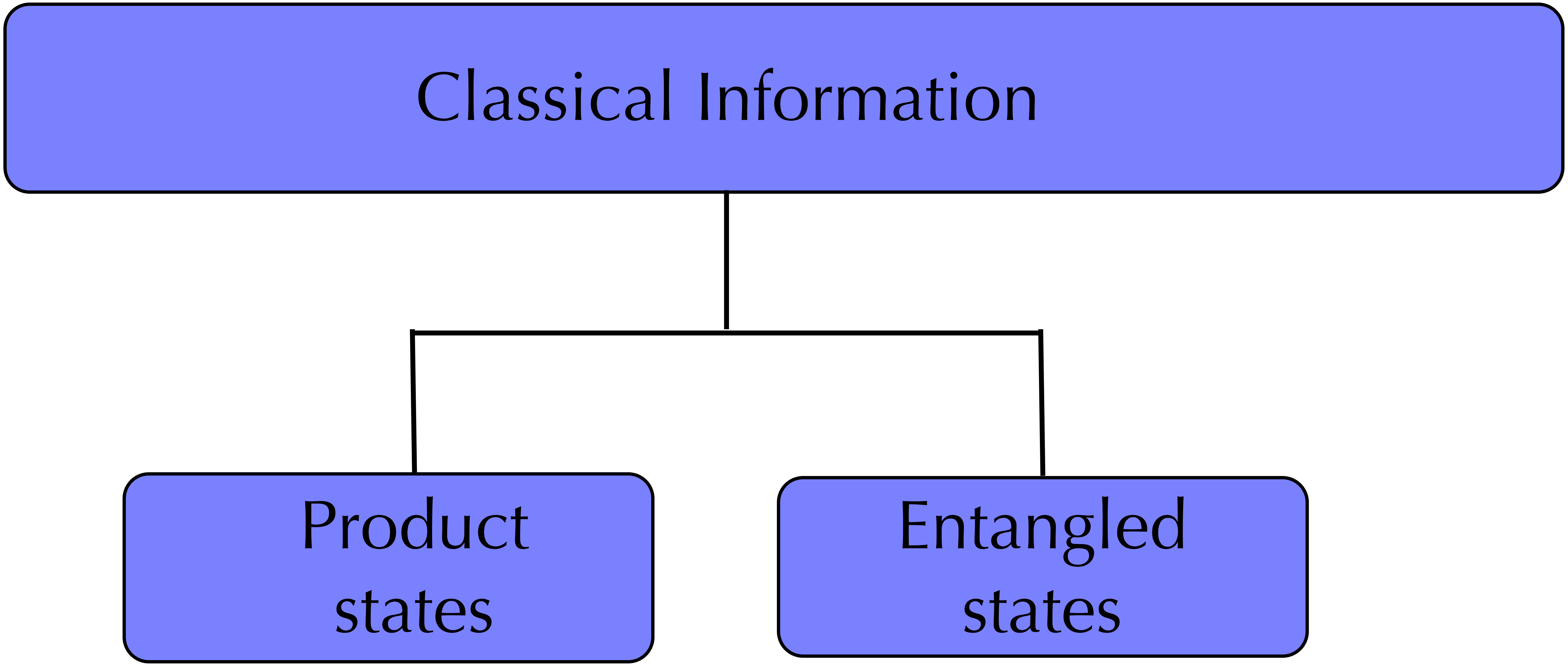


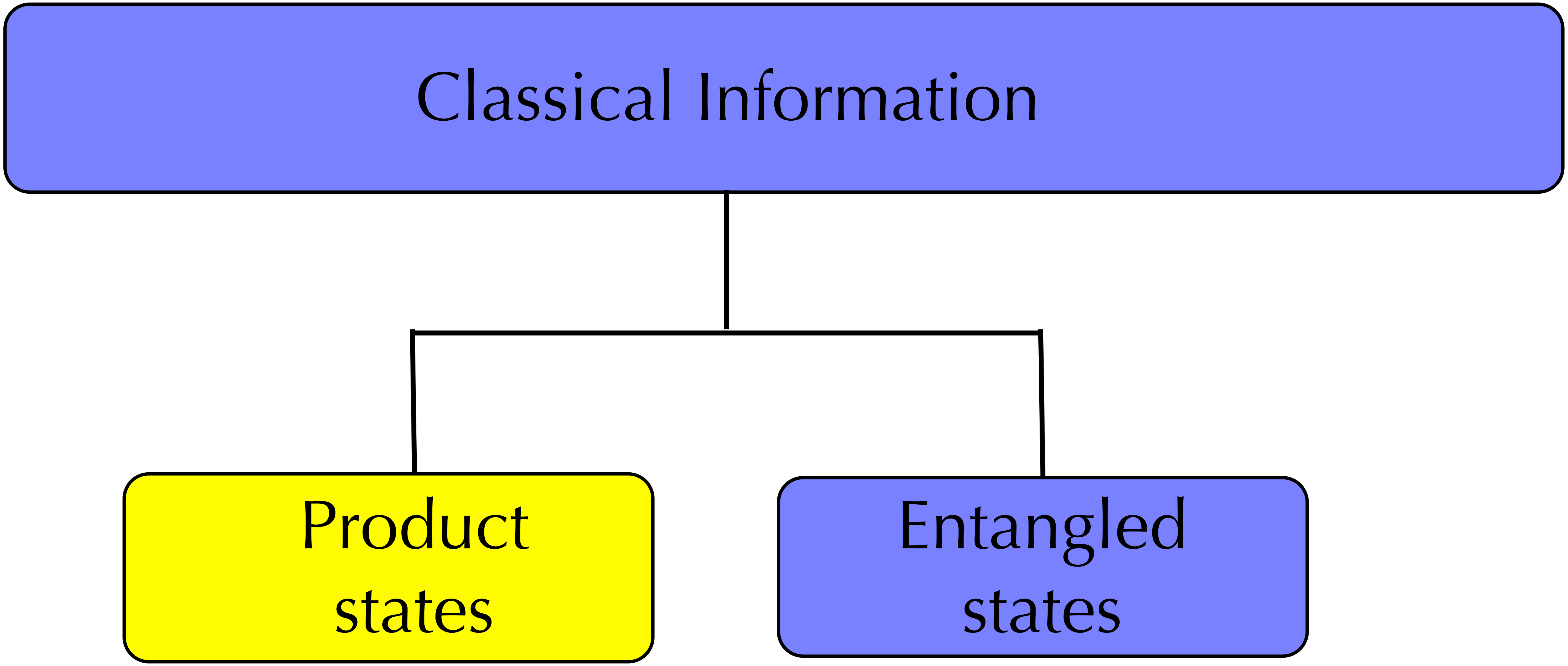
Classical-quantum channel:

$$\Phi : \mathcal{X} \to \mathcal{B}(\mathcal{K})$$

taking each $x \in \mathcal{X}$ to $\Phi_x = \Phi(\rho_x)$

where $\{\rho_x\}_{x \in \mathcal{X}}$ represents the input state to the channel Φ and $\rho_x \in \mathcal{B}(\mathcal{H})$


Classical Information
Product states
Entangled states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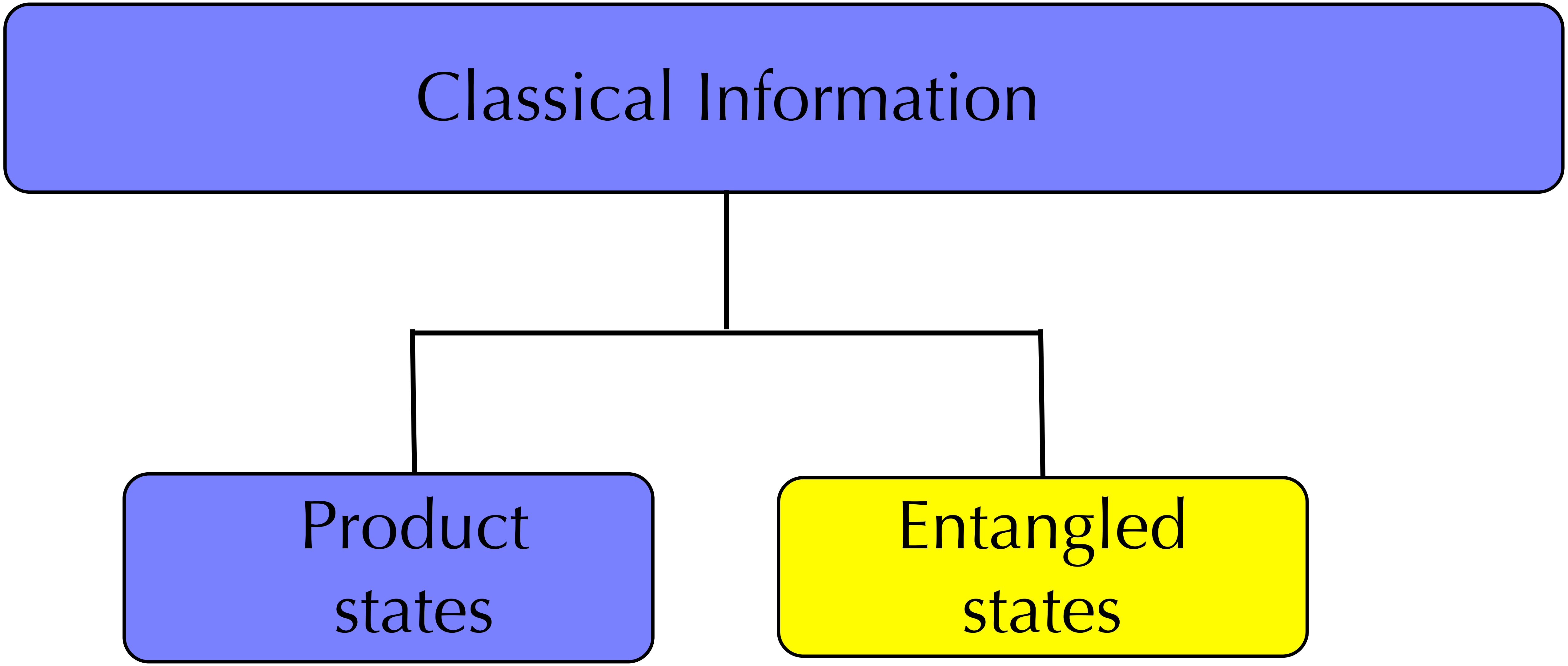
Holevo - Schumacher - Westmoreland

$$\chi^*(\Phi) = \max_{\{p_j,\rho_j\}} \left[S\left(\Phi\left(\sum_j p_j \rho_j \right)\right) - \sum_j p_j S\left(\Phi(\rho_j)\right) \right]$$

Product - state capacity for arbitrary channel Φ

Note: strong converse known to hold

[Ogawa & Nagaoka/ Winter (1999)]


Classical Information
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Classical capacity of a quantum channel

The classical capacity of a quantum channel Φ is given by

$$C\left(\Phi\right) = \lim_{n\to\infty} \frac{1}{n}\chi^{*}\left(\Phi^{(n)}\right)$$

where

$$\chi^{*}(\Phi^{(n)}) = \sup_{\{p_j^{(n)},\rho_j^{(n)}\}}\left[S\left(\Phi^{(n)}\left(\sum_j p_j^{(n)}\rho_j^{(n)}\right)\right) - \sum_j p_j^{(n)} S\left(\Phi^{(n)}\left(\rho_j^{(n)}\right)\right)\right]$$

denotes the Holevo capacity of the channel Φ^{n} with an n fold input state ensemble.

Additivity of the Holevo quantity

The Holevo capacity of a channel Φ is *additive* if the following holds

$$\chi^* (\Phi \otimes \Psi) = \chi^* (\Phi) + \chi^* (\Psi)$$

If Holevo capacity of a channel Φ is additive then

$$\chi^* \left(\Phi^{\otimes n}\right) = n \, \chi^* (\Phi)$$

which implies that the classical capacity of $\Phi^{\otimes n}$ is equal to the product-state capacity:

$$C (\Phi) = \chi^* (\Phi) \, .$$

Implies that the classical capacity of that channel cannot be increased by entangling inputs across two or more uses of the channel

Additivity results

Additivity has been proved for the following channels:

- d - dimensional depolarizing channel
[C. King, IEEE Transactions on Information Theory (2003)]

- Unital qubit channels
[C. King, Journal of Mathematical Physics (2003)]

- Entanglement-breaking channels
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Counter-example : Random Unitary Channel
M.B. Hastings, Nature Physics (2009)

Question: Is this the end of the story?

No!

(at least for quantum Shannon theorists ;-)

Motivation: better understand the non-additivity property of quantum channels

What about other channels?

We consider the qubit amplitude damping channel

- Non- unital
- degradable

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Quantum capacity is known

[Shor & Devetak 2003]

Amplitude damping channel

Kraus operators

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

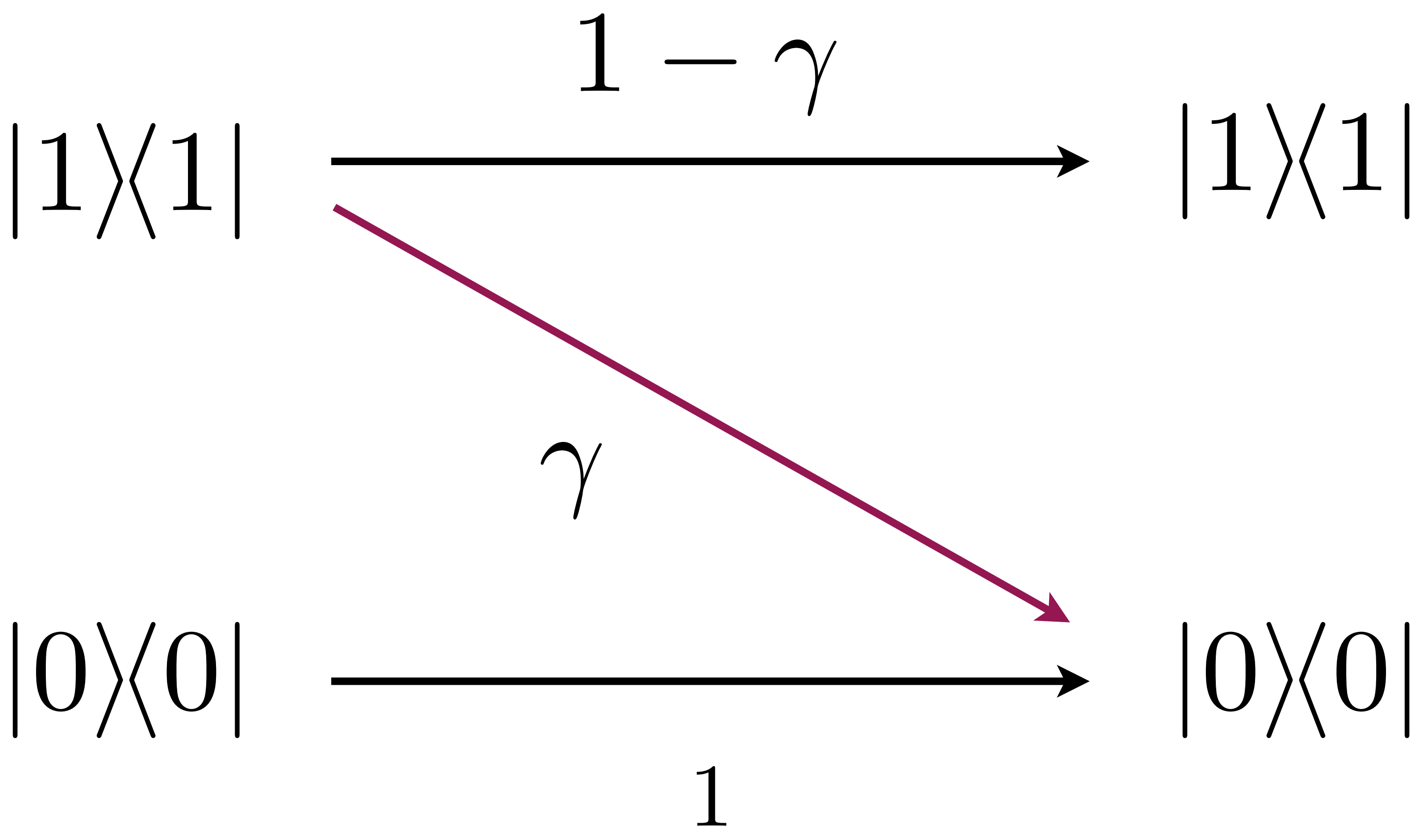
with error parameter $0 \leq \gamma \leq 1$ acting on

$$\rho = \begin{pmatrix} a & b \\ \bar{b} & 1-a \end{pmatrix}$$

can be written as

$$\Phi(\rho) = \begin{pmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ \bar{b}\sqrt{1-\gamma} & (1-a)(1-\gamma) \end{pmatrix}$$

Interpreting the qubit amplitude-damping channel



Calculating the product-state capacity for the amplitude damping channel

$$\chi^*(\Phi) = \max_{\{p_j,\rho_j\}} \left[S\left(\Phi\left(\sum_j p_j \rho_j \right) \right) - \sum_j p_j S\left(\Phi(\rho_j)\right) \right]$$

Trick:

- Replace each state in the ensemble by itself and its antipodal state with half the probability

Result:

- Concave function of single parameter $\longrightarrow$ unique max

Unique maximizer

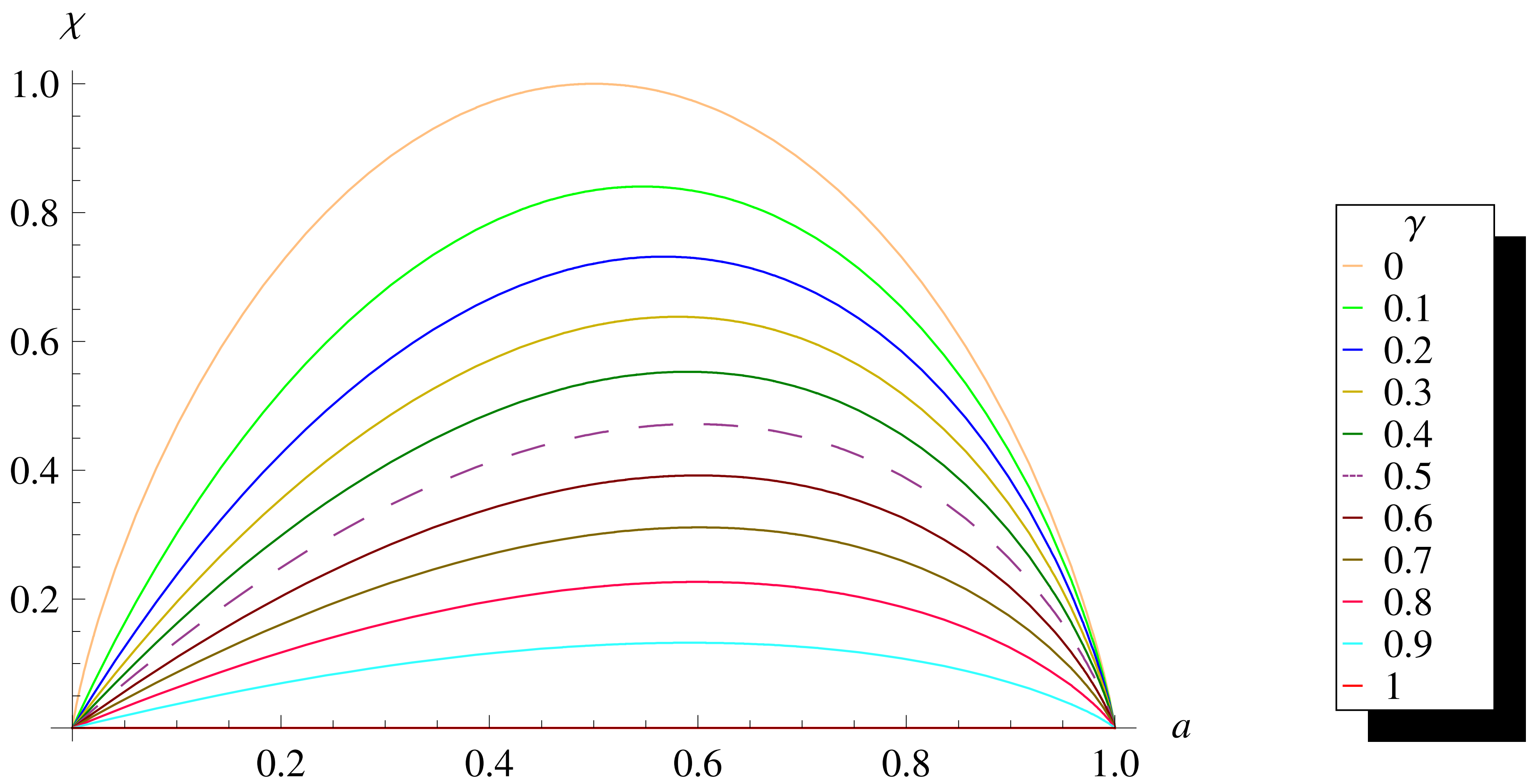
$$\chi^*(\Phi) = \max_{\{p_j,\rho_j\}} \left[S\left(\Phi\left(\sum_j p_j \rho_j\right)\right) - \sum_j p_j S\left(\Phi(\rho_j)\right) \right]$$

Using the replacement trick:

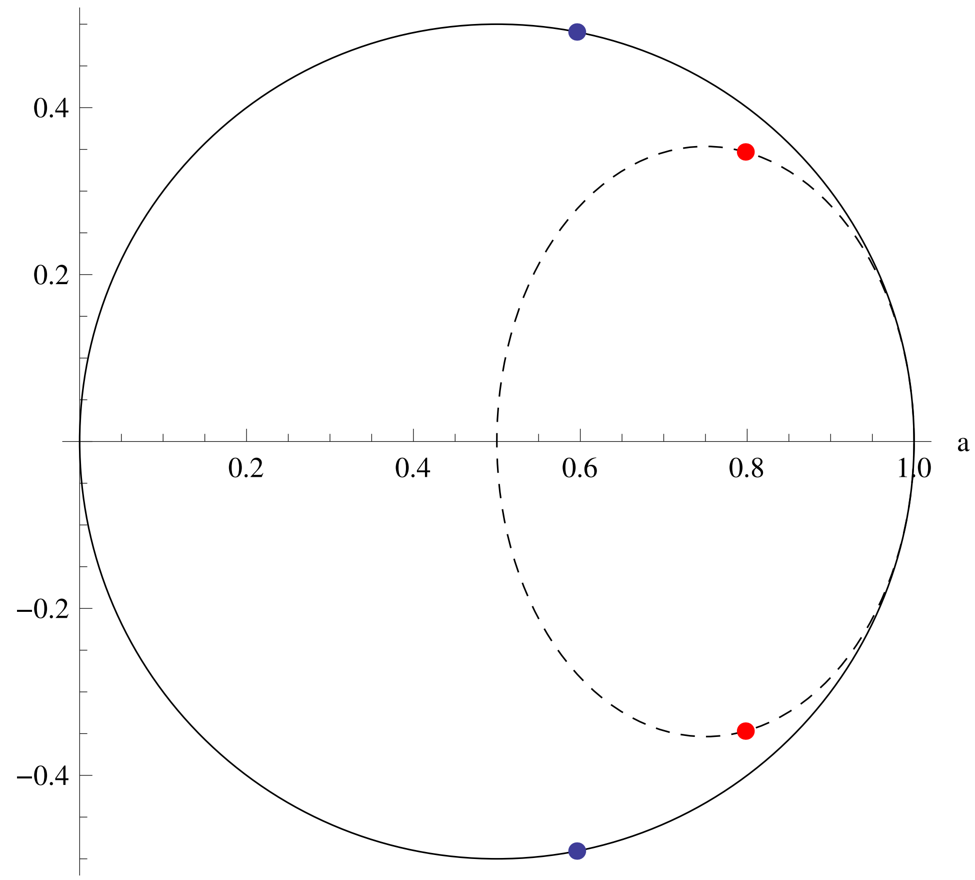
- Expression for $\chi(\Phi)$ in terms of single parameter a
- Simultaneously *max* first term and *min second term*
- Show that second term is *convex* ⟶ overall *concave*
- Unique maximizer

Product state capacity for the amplitude damping channel

Maximizing ensemble: two non-orthogonal states



Example



Optimal input states (blue) to the amplitude-damping channel with $\gamma = 0.5$ and the resulting output states from the channel (red).

Result - joint with Tony Dorlas

Question: Is the Holevo capacity additive?

Try for two channel uses $\Phi \otimes \Phi$

Answer: Yes, for large noise γ

Extend technique for single channel use

leading to a unique maximizer of the form

$$\rho_a \otimes \rho_a$$

Implying additivity of

$$\chi^*(\Phi \otimes \Phi)$$

Conclusion

- Holevo capacity is additive for *two uses* of the qubit amplitude damping channel for small

$$\epsilon = 1 - \gamma$$

- Good: indication of additivity

- Bad: No clear way to proceed

- Ugly: messy calculations

- Bigger picture ... new techniques... one-shot?

Thank you!