Trace-norm contraction under tensor product channels

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joint work (in progress) with:
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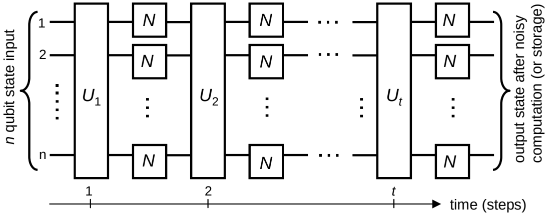
TECHNISCHE UNIVERSITÄT MÜNCHEN

MARIE CURIE ACTIONS

SEVENTH FRAMEWORK PROGRAMME

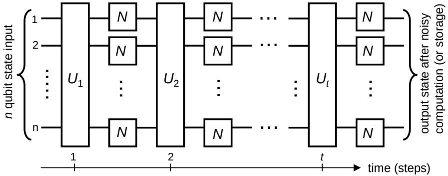
"Quantum Refrigerator"

[Ben-Or, Gottesman, Hassidim arXiv:1301.1995]



- finite size n
- noise $\mathcal{N}^{\otimes n}$
- local unitaries U_i
- **no fresh ancillas** (e.g. [Landauer '61])
  $\rightarrow$ no traditional QEC

"Quantum Refrigerator" model



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Theorem [Ben-Or et al., arXiv:1301.1995]

Let noise $\mathcal{N}$ be *non-unital* and sufficiently weak, acting independently on n physical *qubits*. **Then:**

Can simulate any ideal quantum circuit of size N and depth T if

$$n \geq O(N \,\mathrm{polylog}(NT)) \,.$$

Proof: *(i)* $\mathcal{N}$ contracts towards $\sigma_0 \neq \mathbb{1}/2 \quad \rightarrow$ can cool R qubits.
(ii) data compression: extract $R(1 - S(\sigma_0))$ pure qubits.
(iii) run fault-tolerant protocol. [need $t = O(T \,\mathrm{polylog}(NT))$] □

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Corollary: If $T \leq e^{O(N)}$, then poly overhead ok: $n = O(\mathrm{poly}(N))$.

[Note 1: if no unitaries ($U_i = \mathbb{1}$) $\Rightarrow$ $T_{max} \simeq (\log n)/\mathrm{gap}(\mathcal{N}) \rightarrow$ “cutoff”]
[Note 2: unital $\mathcal{N}$ $\Rightarrow$ $T_{max} \leq (\log n)/\mathrm{gap}(\mathcal{N})$ ($\rightarrow$ Alexander M.-H.)]

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Conjecture [Ben-Or et al.]

The maximal computation time is $T \leq O(e^n)$.

Outline

I. Doeblin-type approach

II. $\mathcal{N}^* \circ \mathcal{N}$ approach

$\rightarrow$ nontrivial contraction result, “single letter”

$\rightarrow$ proof of conjecture

III. Further questions

I. Doeblin-type approach

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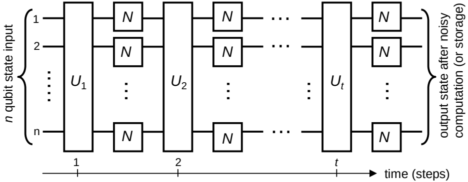

If $\mathcal{N}(\rho) = (1-p)\rho + p\sigma_0 \mathrm{Tr}[\rho]$:

- at *each* time-step:
  $\mathrm{Prob}[n \text{ noise hits}] = p^n$
- expected survival time:
  $T_{max} \sim 1/p^n = O(e^n)$

I. Doeblin-type approach

Assume $\mathcal{N}(\rho) = (1-p)\rho + p\sigma_0 \mathrm{Tr}\,[\rho]$.

Formally: Doeblin method – contraction coefficient η

$$\begin{aligned}
\eta[\mathcal{N}^{\otimes n}] &:= \sup_{\rho,\sigma} \frac{1}{2} \|\mathcal{N}^{\otimes n}(\rho-\sigma)\|_1 \\
&= \sup_{\rho,\sigma} \frac{1}{2} \left\| \big((1-p)\mathrm{id} + p\sigma_0 \mathrm{Tr}\big)^{\otimes n}(\rho-\sigma) \right\|_1 \\
&= \sup_{\rho,\sigma} \frac{1}{2} \Big\| \underbrace{(\ldots\ldots)}_{\text{mass} = (1-p^n)} + \underbrace{p^n \sigma_0^n \mathrm{Tr}[\rho-\sigma]}_{\text{mass} = p^n} \Big\|_1 \leq 1-p^n
\end{aligned}$$

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$$= \sup_{\rho,\sigma} \frac{1}{2} \| \underbrace{(\ldots\ldots)}_{\text{mass}=(1-p^n)} + \underbrace{p^n \sigma_0^n \mathrm{Tr}[\rho-\sigma]}_{\text{mass}=p^n} \|_1 \leq 1-p^n$$

$$\Rightarrow \frac{1}{2} \| \underbrace{\mathcal{N}^{\otimes n} \circ U_t \circ \ldots \mathcal{N}^{\otimes n} \circ U_1}_{\text{noisy circuit; any q-channels}}(\rho-\sigma) \|_1 \leq \left(\eta[\mathcal{N}^{\otimes n}]\right)^t \leq (1-p^n)^t$$

Thus: bias $\leq \varepsilon$ for $t \geq T_{max} := \frac{\log \varepsilon}{\log(1-p^n)} \simeq \frac{1}{p^n} = O(e^n)$. ☺

I. Doeblin-type approach

$\rightarrow$ need $\mathcal{N}(\rho) = (1-p)\widetilde{\mathcal{N}}(\rho) \; + \underbrace{p\sigma_0 \mathrm{Tr}\,[\rho]}_{\text{"forgetful piece"}}$

"max forgetful piece": $v_{opt}(\mathcal{N}) :=$ maximize $\mathrm{tr}[P]$
s.th. $P \otimes \mathbb{1} \leq \mathrm{Choi}_{\mathcal{N}}$.

$$\Rightarrow \quad T_{max} \simeq 1/[v_{opt}(\mathcal{N})]^n \;\sim\; O(e^n)$$

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But: extremal channels, e.g. $\mathcal{N}_{AD}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} := \begin{pmatrix} a+pc & b\sqrt{1-p} \\ b^*\sqrt{1-p} & c(1-p) \end{pmatrix}$

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Cannot improve by tensor products $\mathcal{N}^{\otimes n} = \left(\mathcal{N}^{\otimes m}\right)^{\otimes (n/m)}$, since:

$$\forall \mathcal{N} \, \forall m : \qquad v_{opt}(\mathcal{N}^{\otimes m}) \; = \; v_{opt}(\mathcal{N})^m \,.$$

[Proof: Strong SDP duality.]

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$\mathcal{N} = \mathcal{N}_{AD}$: $\lambda_{min}[\mathrm{Choi}_{\mathcal{N}_{AD}^*\mathcal{N}_{AD}}] \geq \frac{p^2}{2}$

$\Rightarrow$ $\left(\eta[\mathcal{N}^{\otimes n}]\right)^t \leq \varepsilon$ for $t \geq T_{max} \simeq \left(\frac{2}{p^2}\right)^n \log\frac{1}{\varepsilon} \sim O(e^n)$

$\to$ runtime bound even for *pure* fixed point

in general: $T_{max} \simeq \frac{\log(1/\varepsilon)}{(\lambda_{min}[\mathrm{Choi}_{\mathcal{N}^*\mathcal{N}}])^n}$

Remark: If $v_{opt}(\mathcal{N}) > 0$, then $\lambda_{min}[\mathrm{Choi}_{\mathcal{N}^*\mathcal{N}}] > 0$

Next: proof of Theorem

II. Tensor-stable contraction from $\mathcal{N}^* \circ \mathcal{N}$

$$\eta[\mathcal{N}^{\otimes n}] = \sup_{\rho,\sigma} \frac{1}{2} \left\| \mathcal{N}^{\otimes n}(\rho) - \mathcal{N}^{\otimes n}(\sigma) \right\|_1$$

$$\leq \sup_{\rho,\sigma} \sqrt{1 - \mathrm{Tr}\left[\mathcal{N}^{\otimes n}(\rho)\mathcal{N}^{\otimes n}(\sigma)\right]} \qquad \text{[Audenaert 2014]}$$

$$= \sqrt{1 - \inf_{\rho,\sigma} \mathrm{Tr}\left[(\mathcal{N}^*\mathcal{N})^{\otimes n}(\rho)\,\sigma\right]}$$

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$$= \sqrt{1 - \inf_{\rho} \lambda_{min}\left[(\mathcal{N}^*\mathcal{N})^{\otimes n}(\rho)\right]}$$

$$= \sqrt{1 - \inf_{\rho} \lambda_{min}\Big[\big(\underbrace{\mathcal{N}^*\mathcal{N} - \mathbb{1}\lambda_{min}[\mathrm{Choi}]\mathrm{Tr}}_{\text{completely positive}} + \underbrace{\mathbb{1}\lambda_{min}[\mathrm{Choi}]\mathrm{Tr}}_{\text{like Doeblin!}}\big)^{\otimes n}(\rho)\Big]}$$

$$\leq \sqrt{1 - \inf_{\rho} \lambda_{min}\Big[\underbrace{\left(\mathbb{1}\lambda_{min}[\mathrm{Choi}_{\mathcal{N}^*\mathcal{N}}]\mathrm{Tr}\right)^{\otimes n}(\rho)}_{\text{independent of } \rho}\Big]}\,. \qquad \square$$

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The [Ben-Or *et al.*] conjecture is on *qubits*.

Theorem:

Let $\mathcal{N} : \mathcal{M}_2 \to \mathcal{M}_2$ be a *qubit* quantum channel.

Then: $$\lambda_{min}[\mathrm{Choi}_{\mathcal{N}^*\mathcal{N}}] \geq \tfrac{1}{8}\mathrm{Tr}\left[(\mathcal{N}(\mathbb{1}) - \mathbb{1})^2\right] .$$

Proof:
- $\lambda_{min} \geq \frac{1}{4}\mathrm{Tr}[\ldots]$ for extremal $\mathcal{N}$.
- $\mathcal{N} = \frac{1}{2}\mathcal{N}_{1,ext} + \frac{1}{2}\mathcal{N}_{1,ext}$.

□

Proves conjecture: RHS > 0 for *non-unital* qubit channels.

III. Summary & Further questions

Here: "Single-letter" trace-norm contraction bounds for $\mathcal{N}^n$.

- $1 - \eta[\mathcal{N}^n] \;\leq\; \inf_{D,E} \left\| D \circ \mathcal{N}^{\otimes n} \circ E \;-\; \mathrm{id}_2 \right\|_{1-1} \;\leq\; 2(1 - \eta[\mathcal{N}^n])$.

  $\rightarrow$ extend m classical bits, $\mathrm{id}_2^{\otimes m}$? multi-state Chernoff ?

- matching bounds for AD-noise?
  - $T_{max} \;\leq\; (2/p^2)^n$ (see above)
  - $T_{max} \;\geq\; 1/p^n$ (by q-Chernoff Theorem)

- q-error-correction: extend to *small* noises: $\|\mathcal{N} - \mathrm{id}\|_\diamond < \delta$

  (not necessarily $\mathcal{N} = (1-p)\mathrm{id} + p\mathrm{Tr}$)

- stronger bounds on T_{max} for (local) unitary actions ?

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Thanks!