# Graphs, classical channels and nonlocality: the interplay

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# Overview

- **Quantum Graph Parameters**
  - Quantum Independence Number
- **Applications in Zero-error Information Theory**
  - Entanglement-assisted Classical Capacity
- **Applications in Nonlocality**
  - Bounds on the value of Nonlocal games

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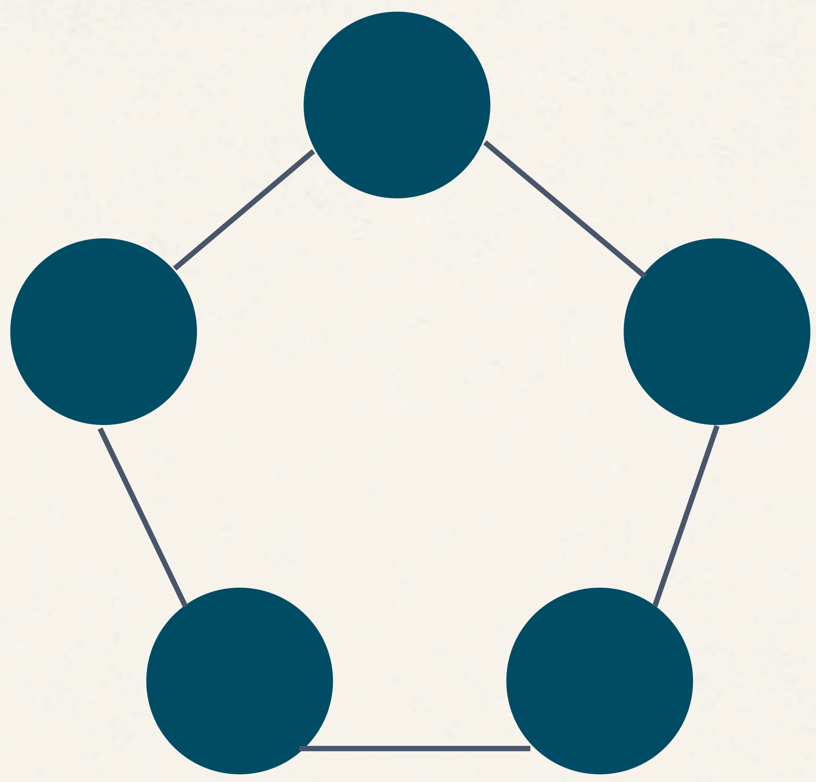

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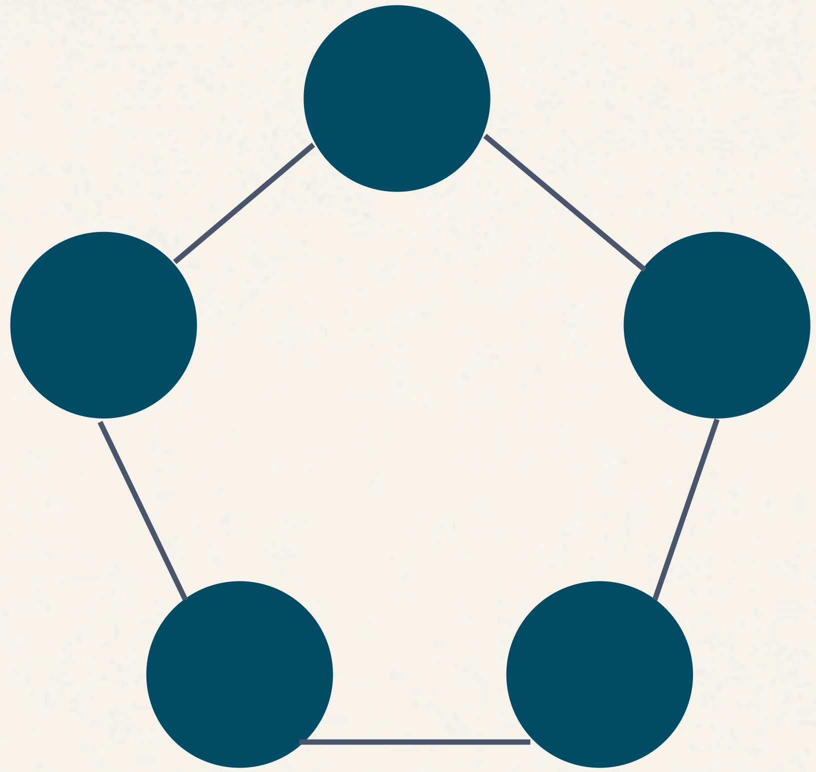

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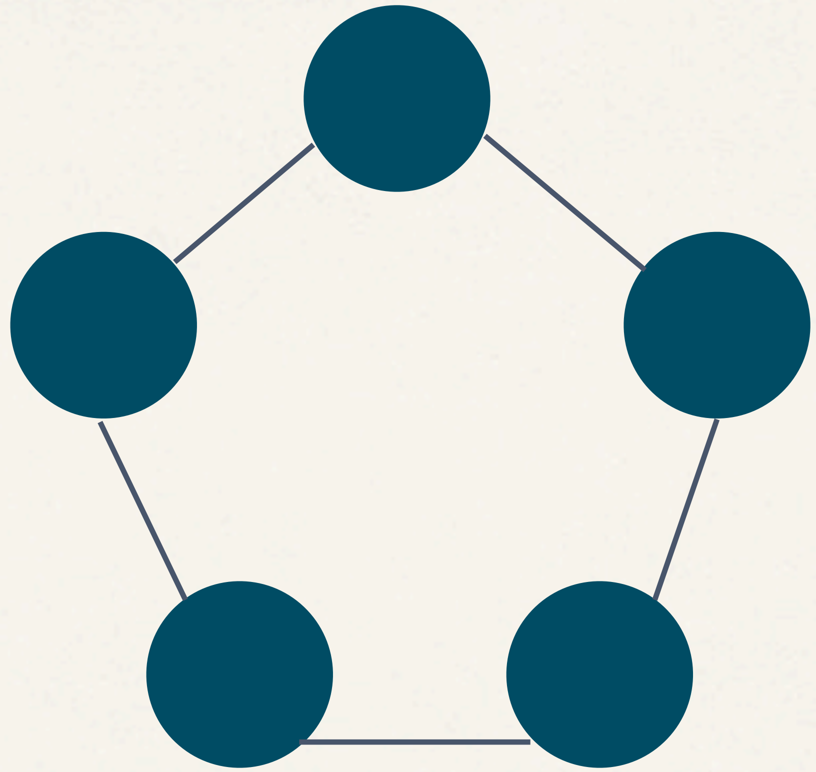

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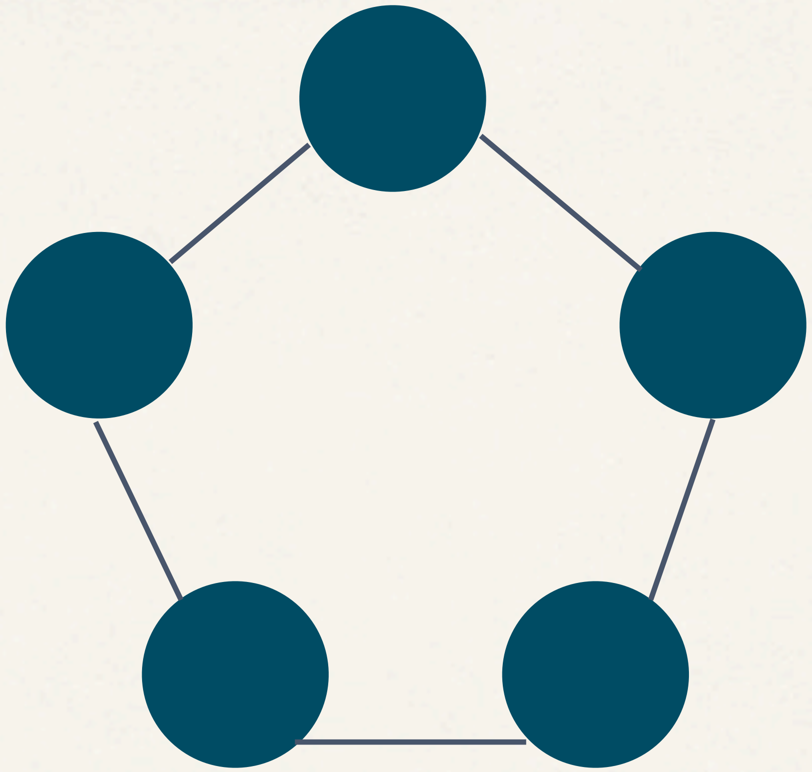

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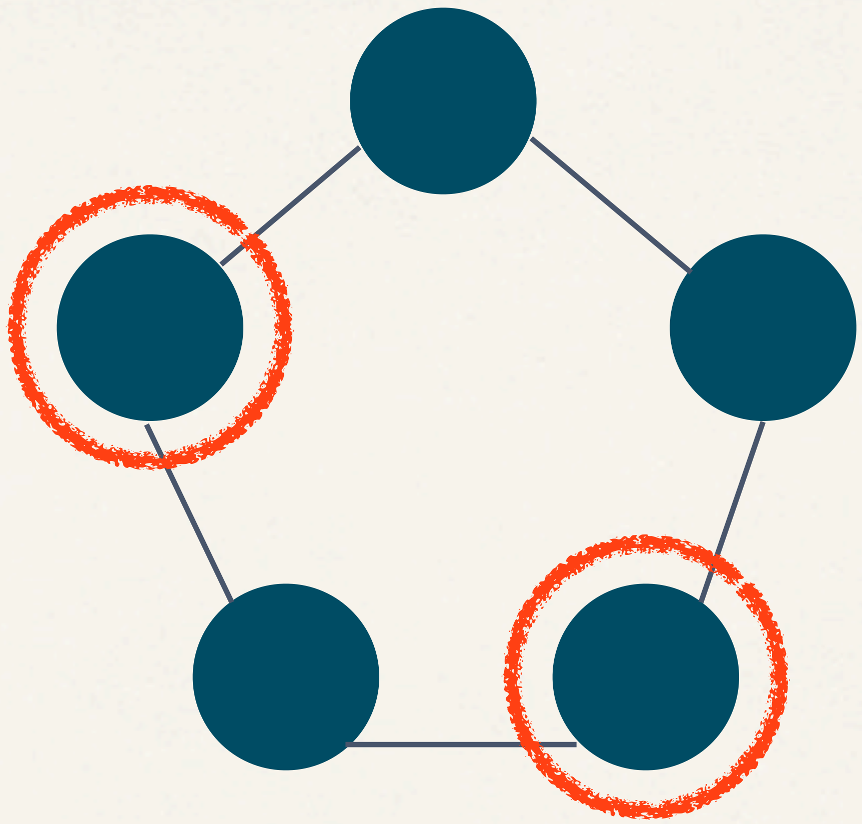

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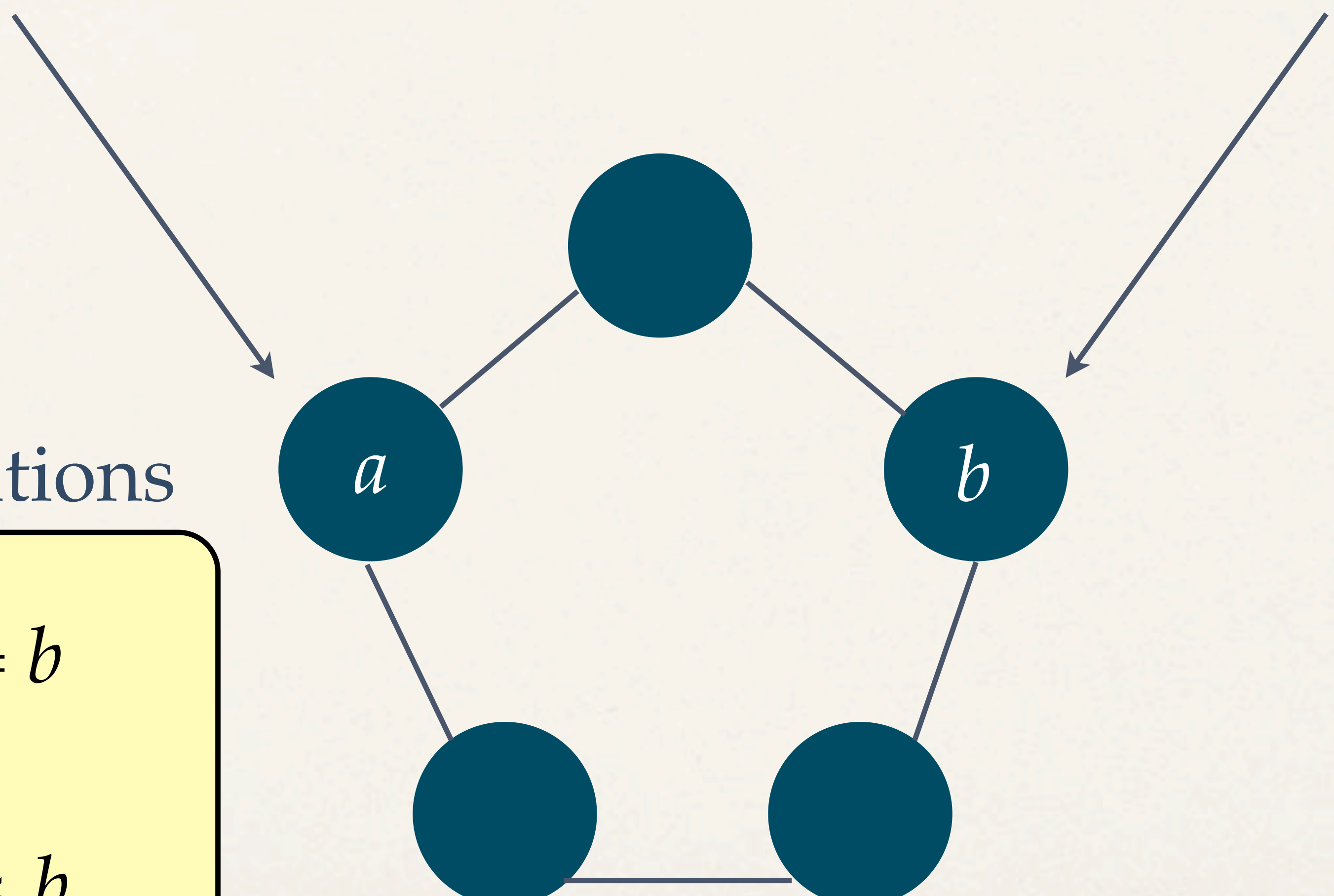
# *t*-independent set game

$i$-th element?

$j$-th element?




Winning Conditions

$$i = j \Rightarrow a = b$$

$$i \neq j \Rightarrow a \neq b$$

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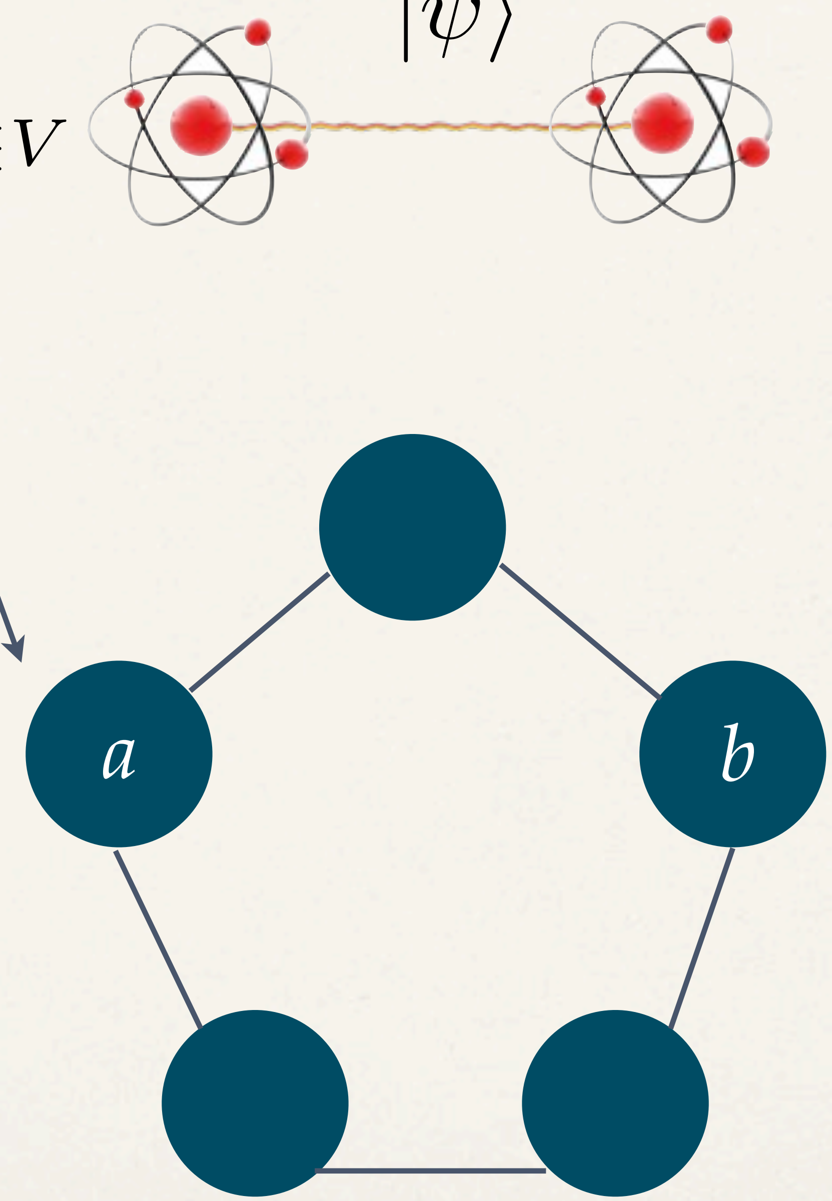
- What about *quantum* players?
- <u>Def</u>. $\alpha_q(G)$ : max $t$ s.t. *quantum* players win with prob. 1

# *t*-independent set game

*i*-th element? *j*-th element?

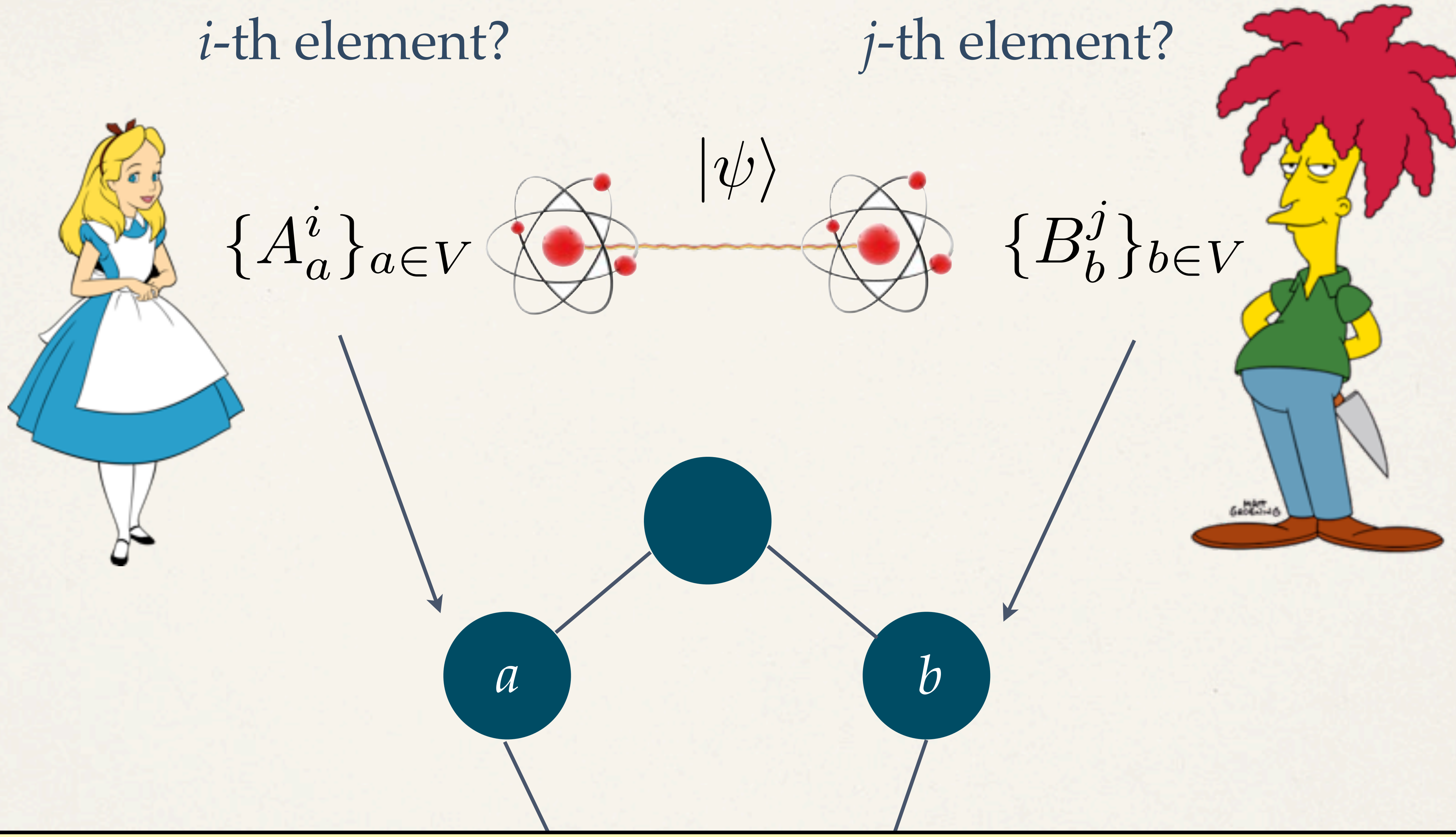
$\{A_a^i\}_{a\in V}$

$|\psi\rangle$




$\{B_b^j\}_{b\in V}$

# *t*-independent set game




Win iff $\langle\psi|A^i_a \otimes B^j_b|\psi\rangle = 0$ whenever

$i = j$ and $a \neq b$

or

$i \neq j$ and $a \simeq b$

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- A **quantum *t*-independent set** is a winning strategy
- Turns out that:
  - A&B use same projective measurements $\{P_u^i\}$ on *maximally entangled state* w.l.o.g.
  - Strategy consists of *t* proj. measurements with outputs in *V* such that:

$$P_u^i \perp P_v^j \quad \text{whenever} \quad i \neq j \quad \textit{and} \quad u \simeq v$$

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- Famous class: Hadamard graphs
- Can be obtained from quantum coloring games [Mančinska, Scarpa, Severini '13]
- Can be obtained starting from any graph [Cabello, Parker, Scarpa, Severini '13]

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HOW MANY COLORS CAN ALICE SELECT ***WITHOUT RISK OF CONFUSION ON BOB'S SIDE***?

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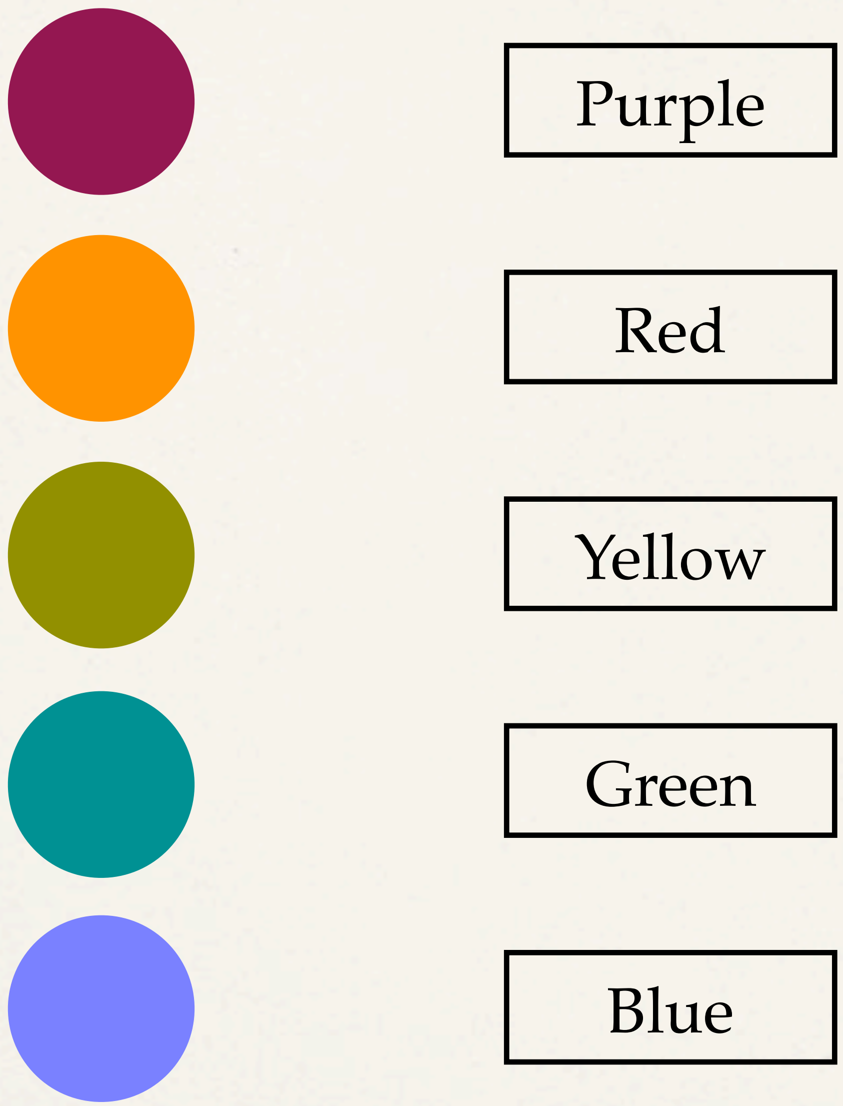

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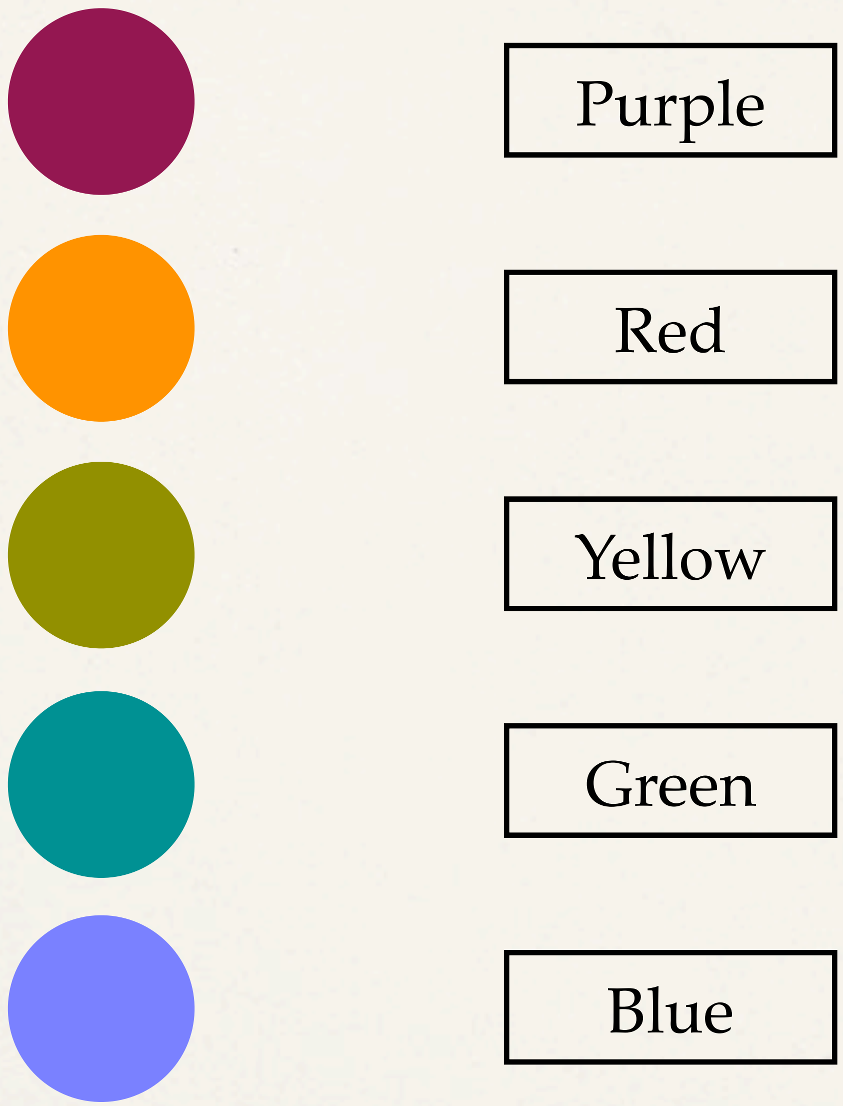

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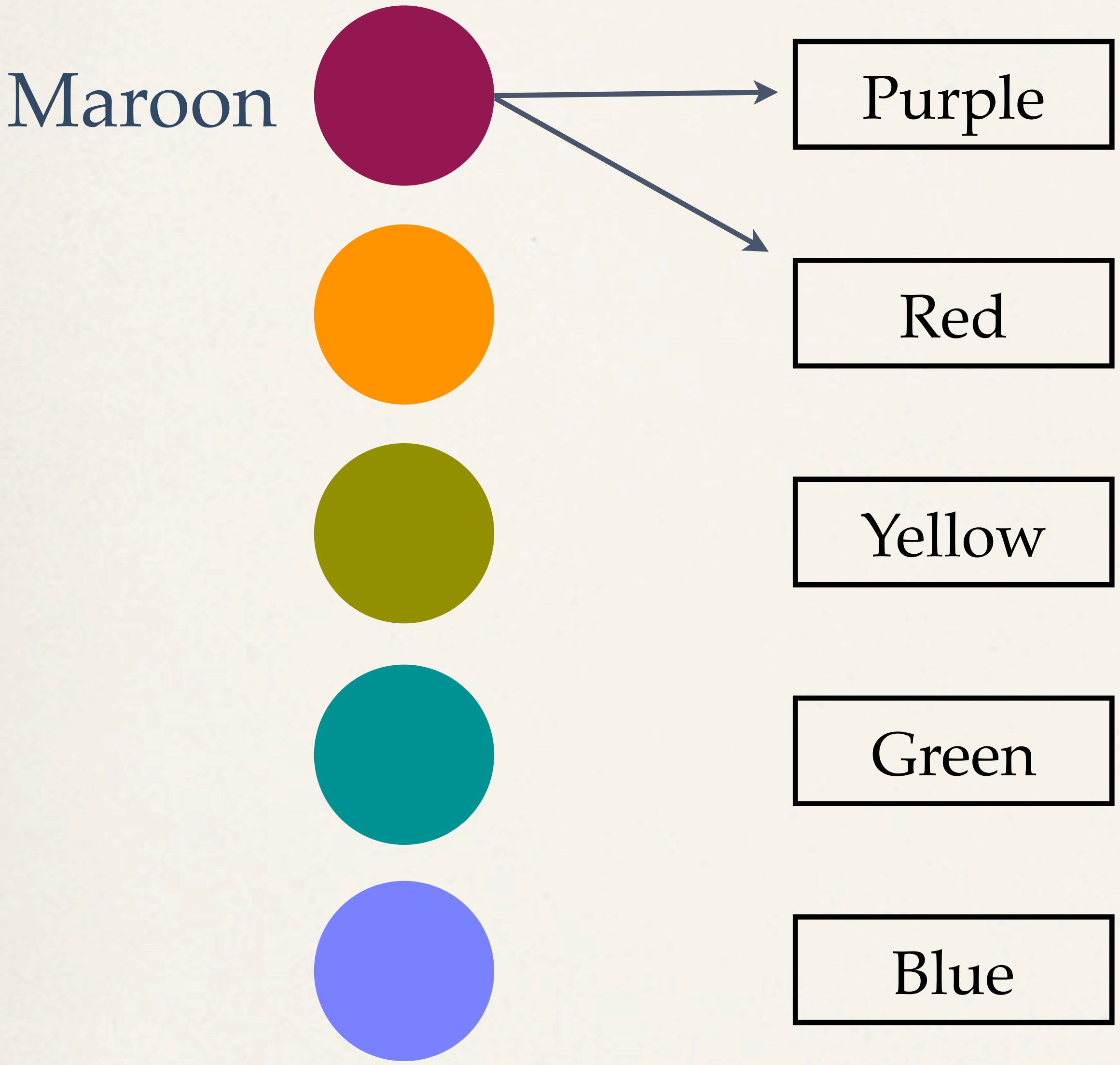

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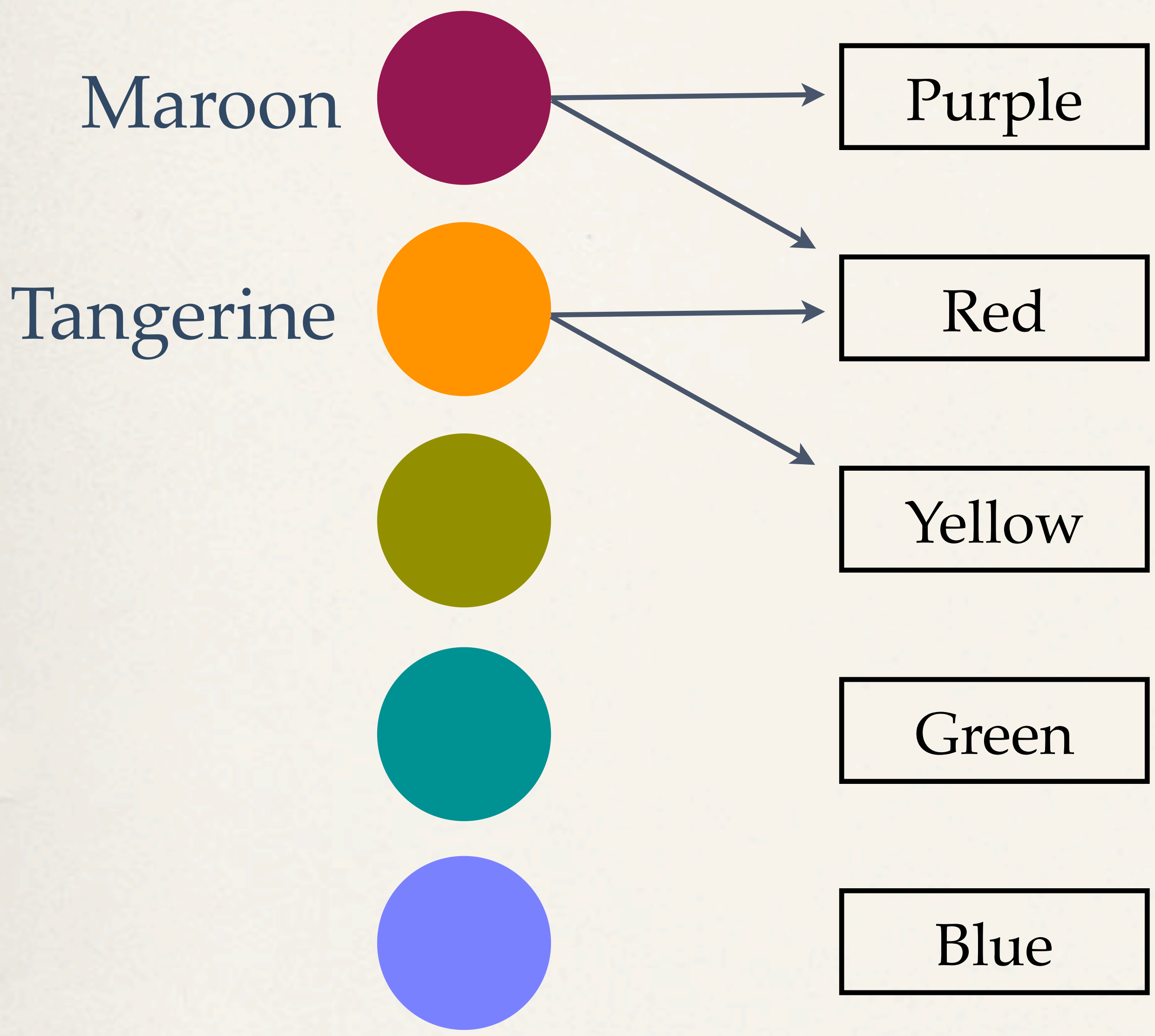

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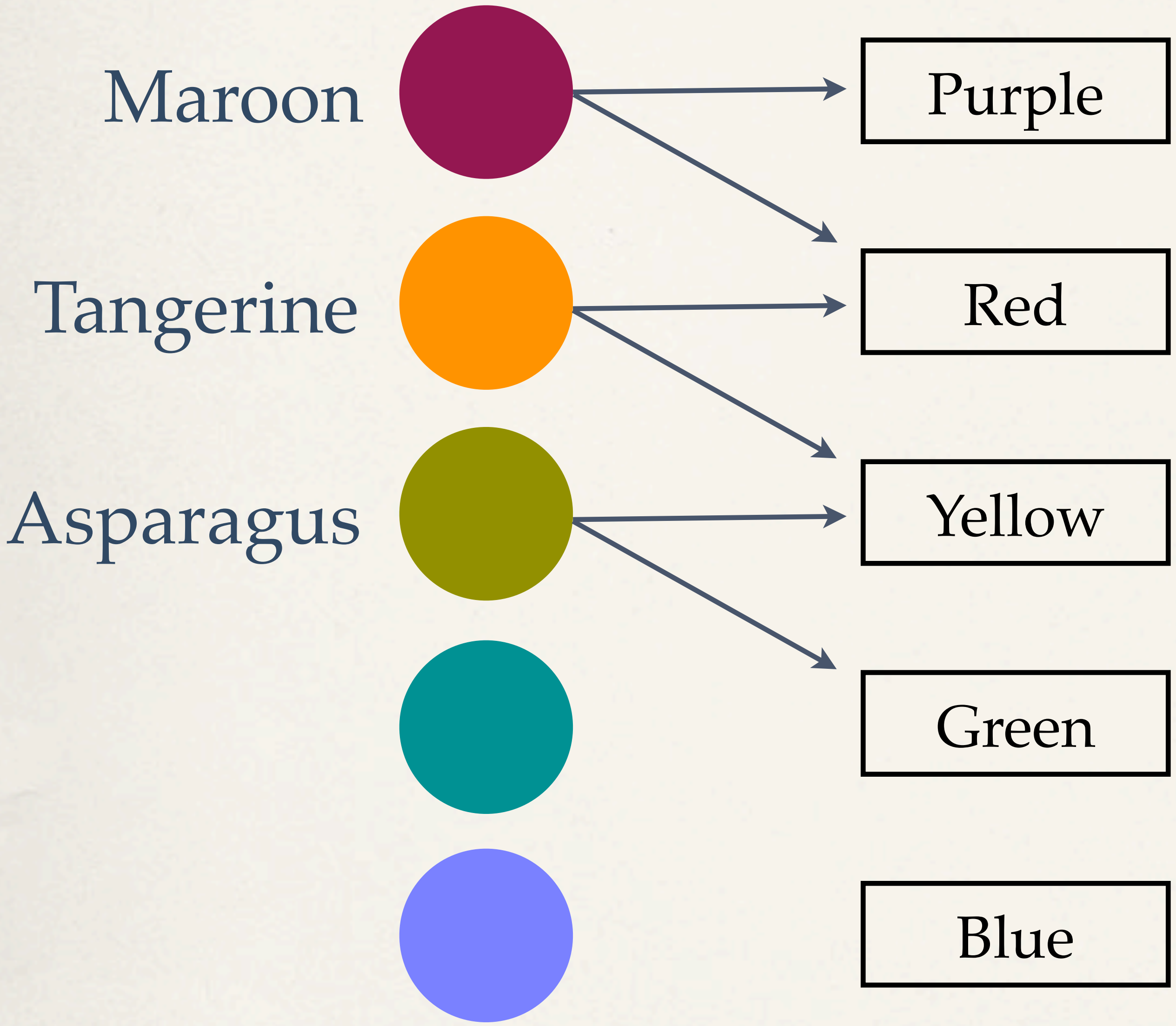

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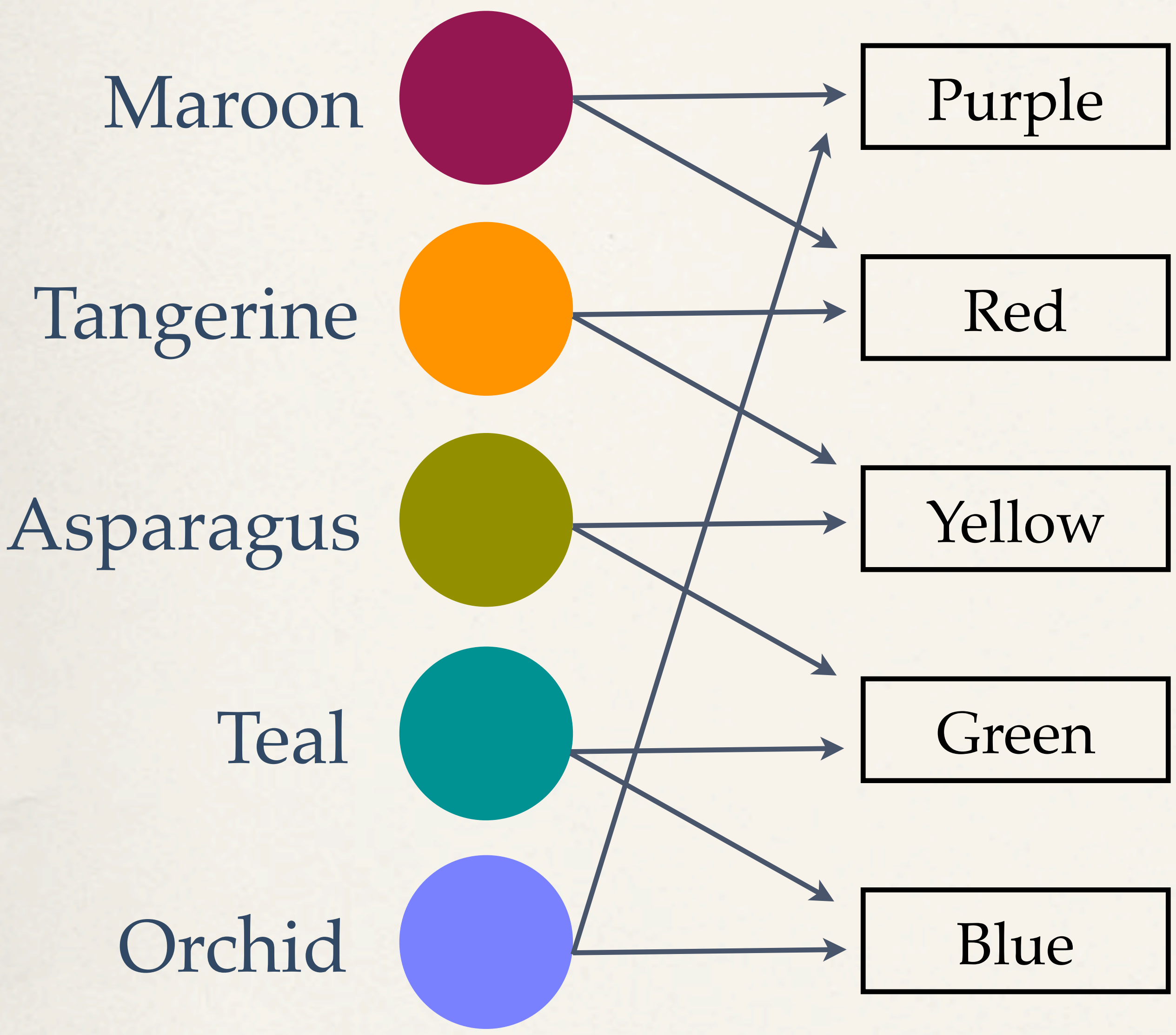

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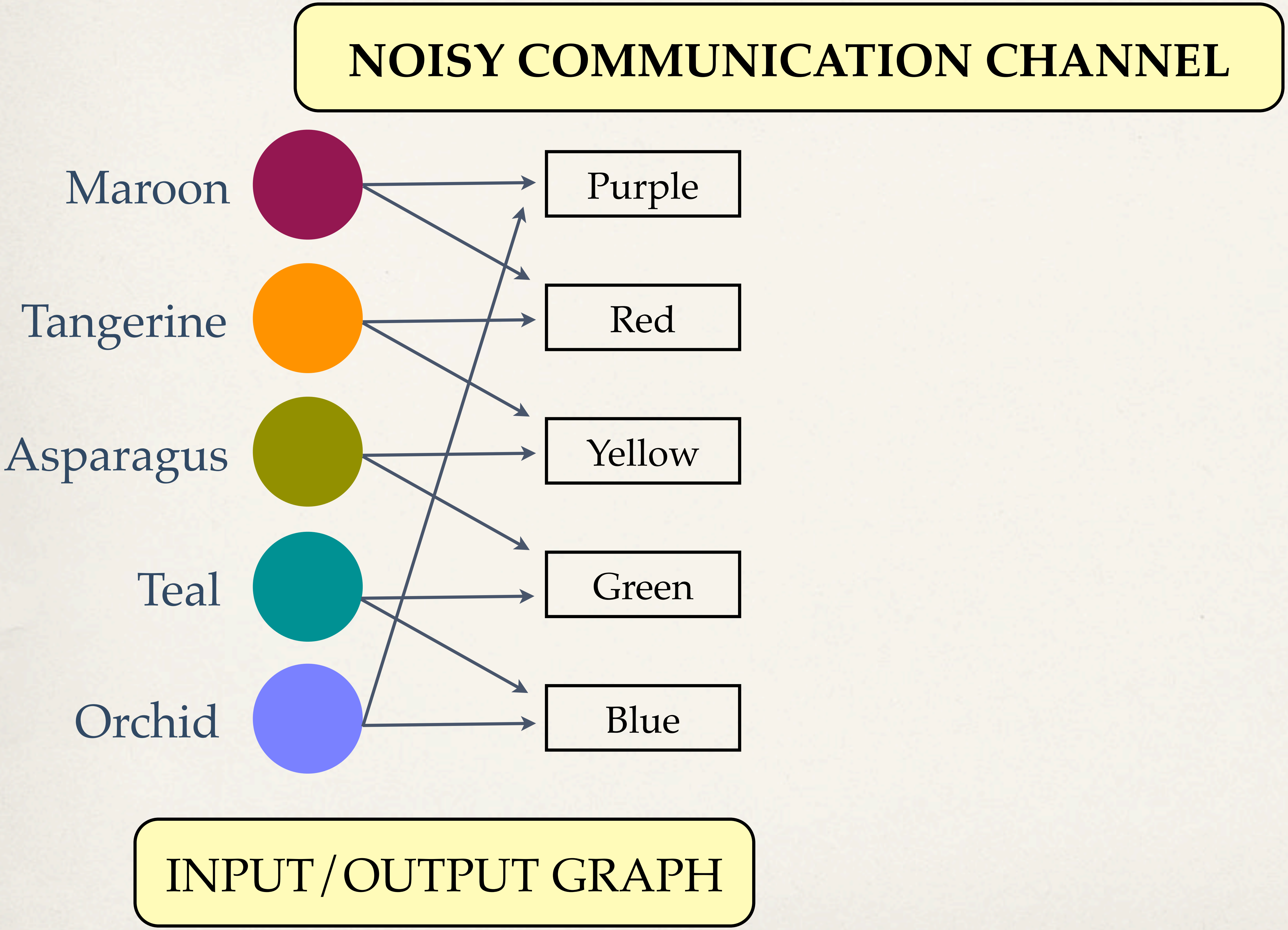

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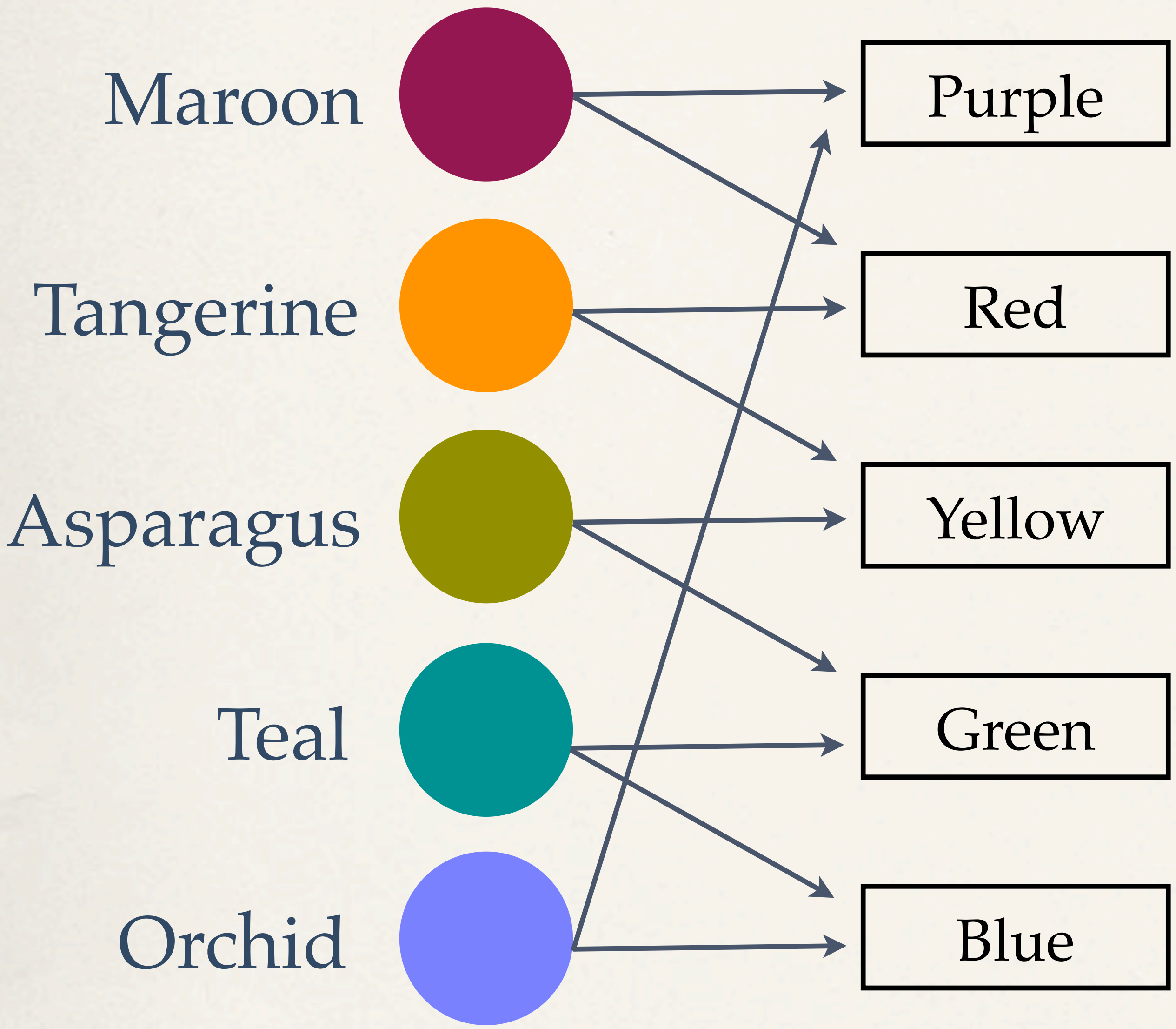
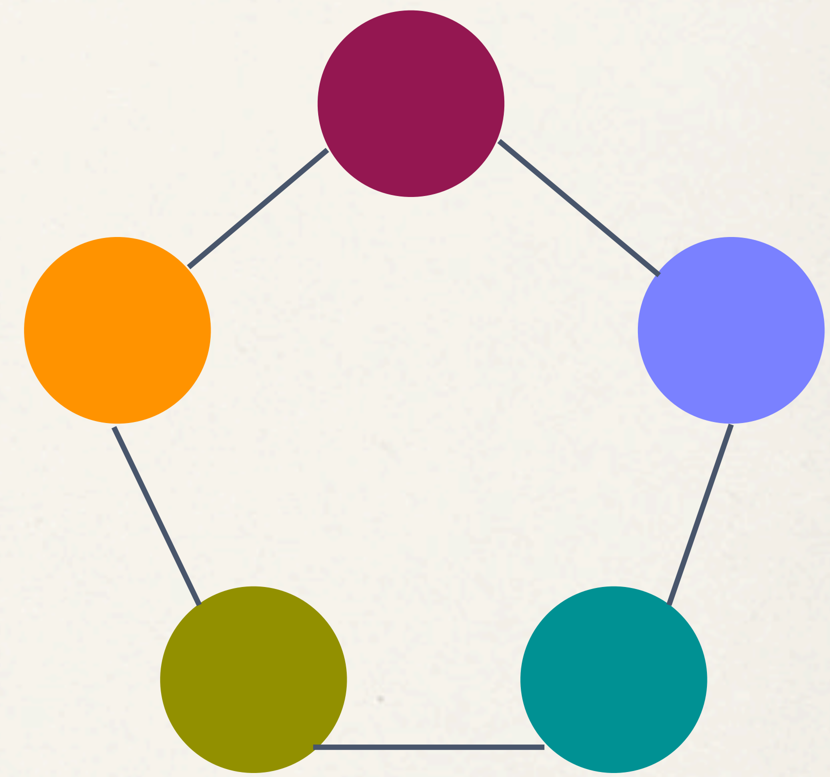
**NOISY COMMUNICATION CHANNEL**




INPUT/OUTPUT GRAPH



CONFUSABILITY GRAPH

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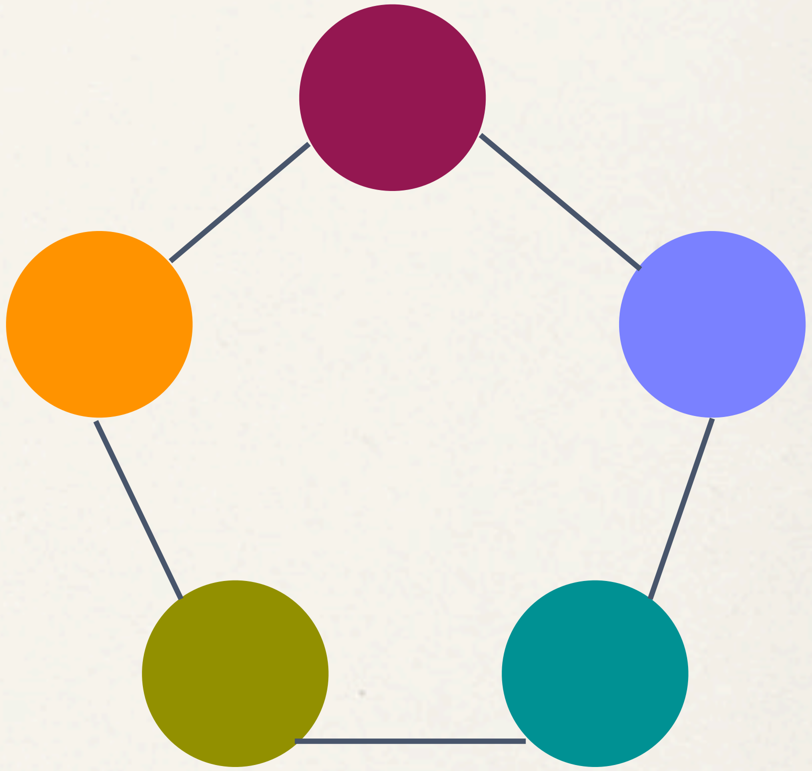

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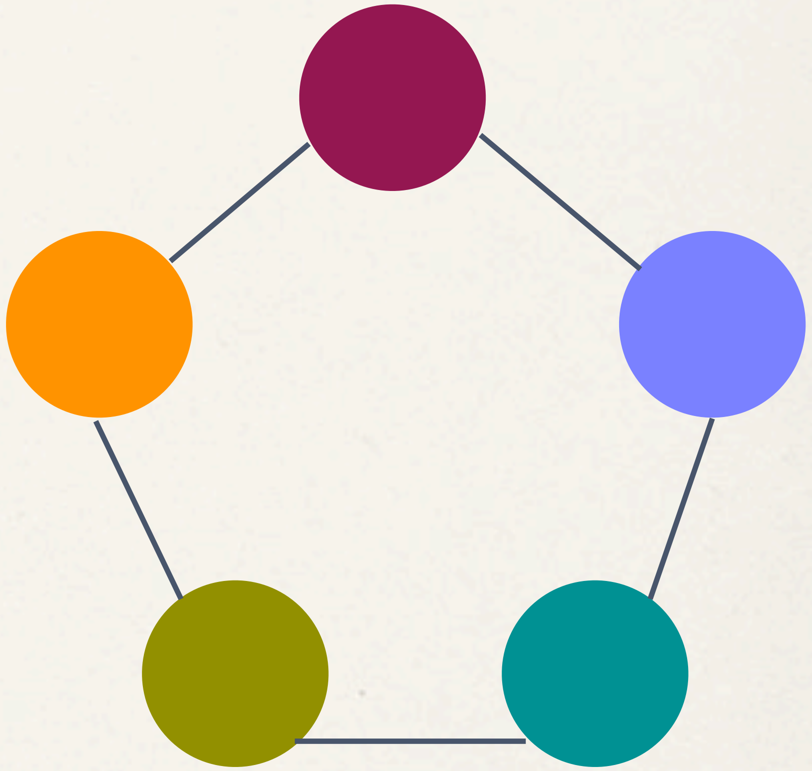

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HOW MANY
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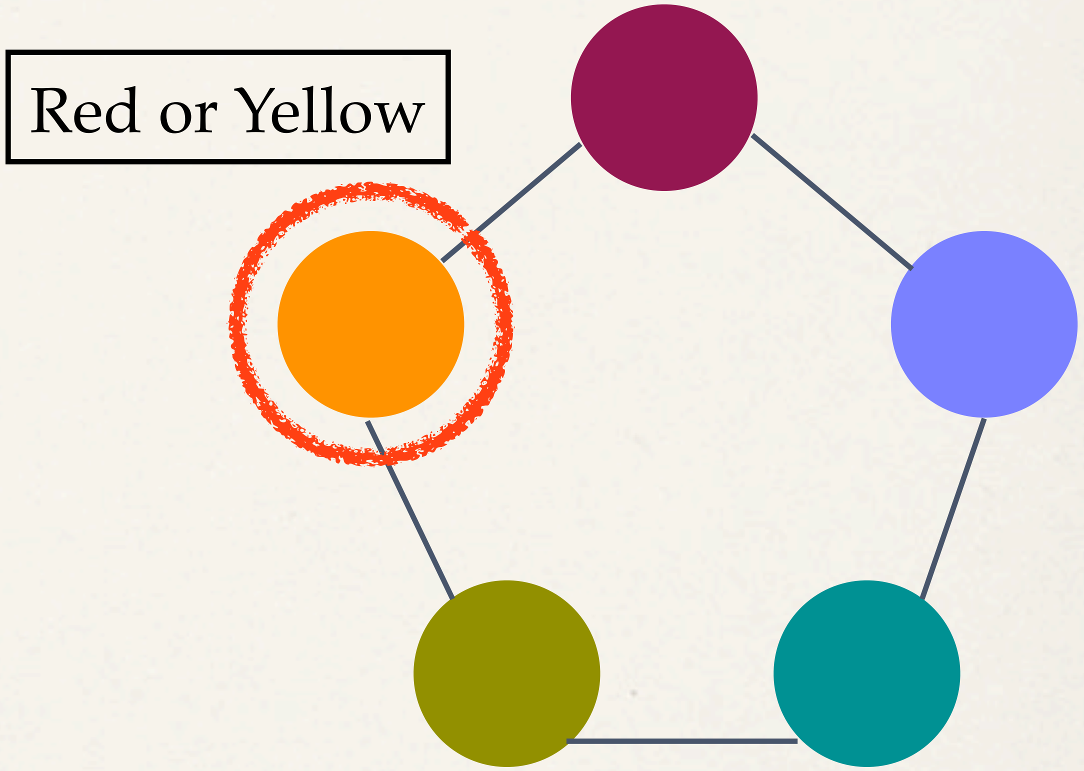


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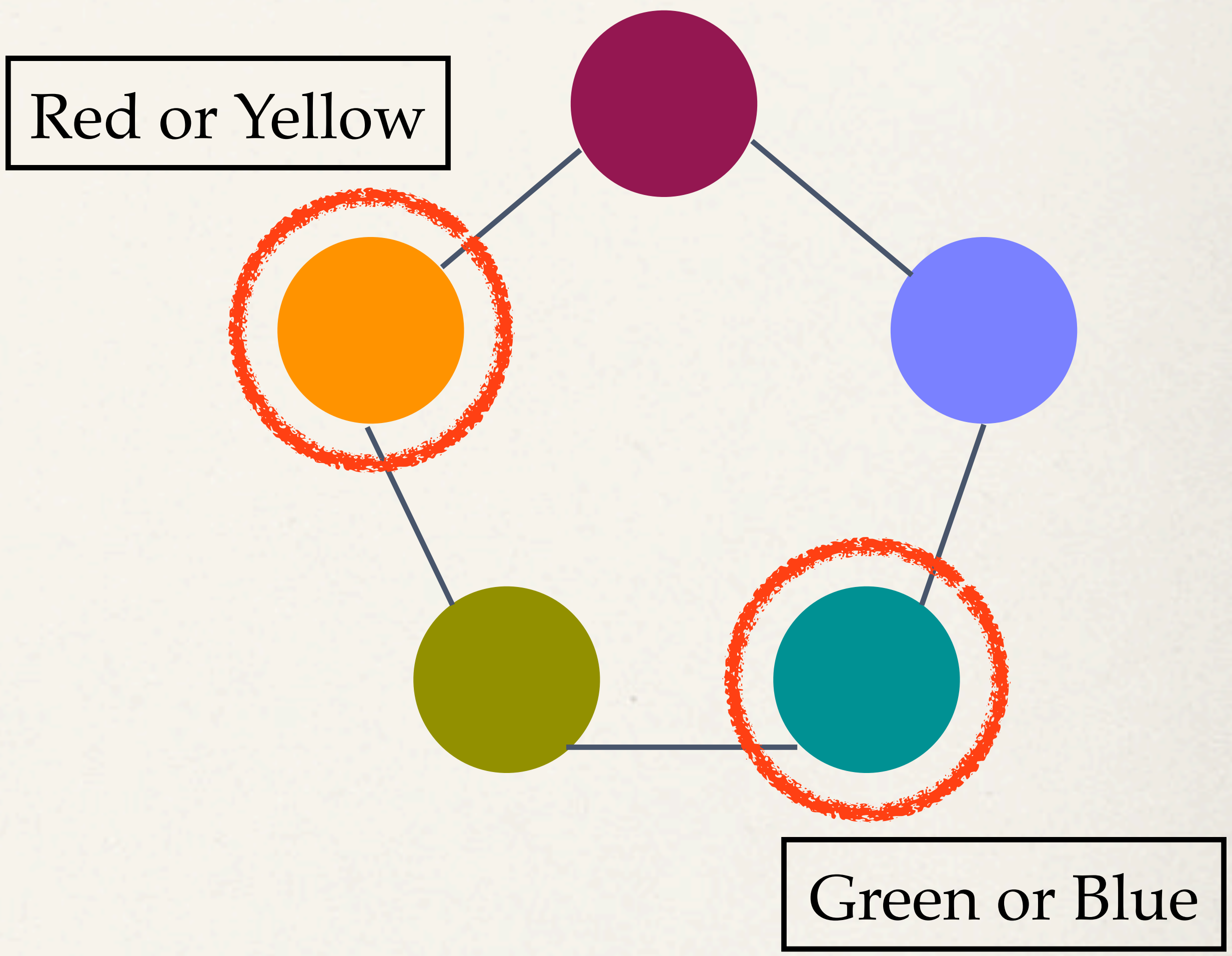


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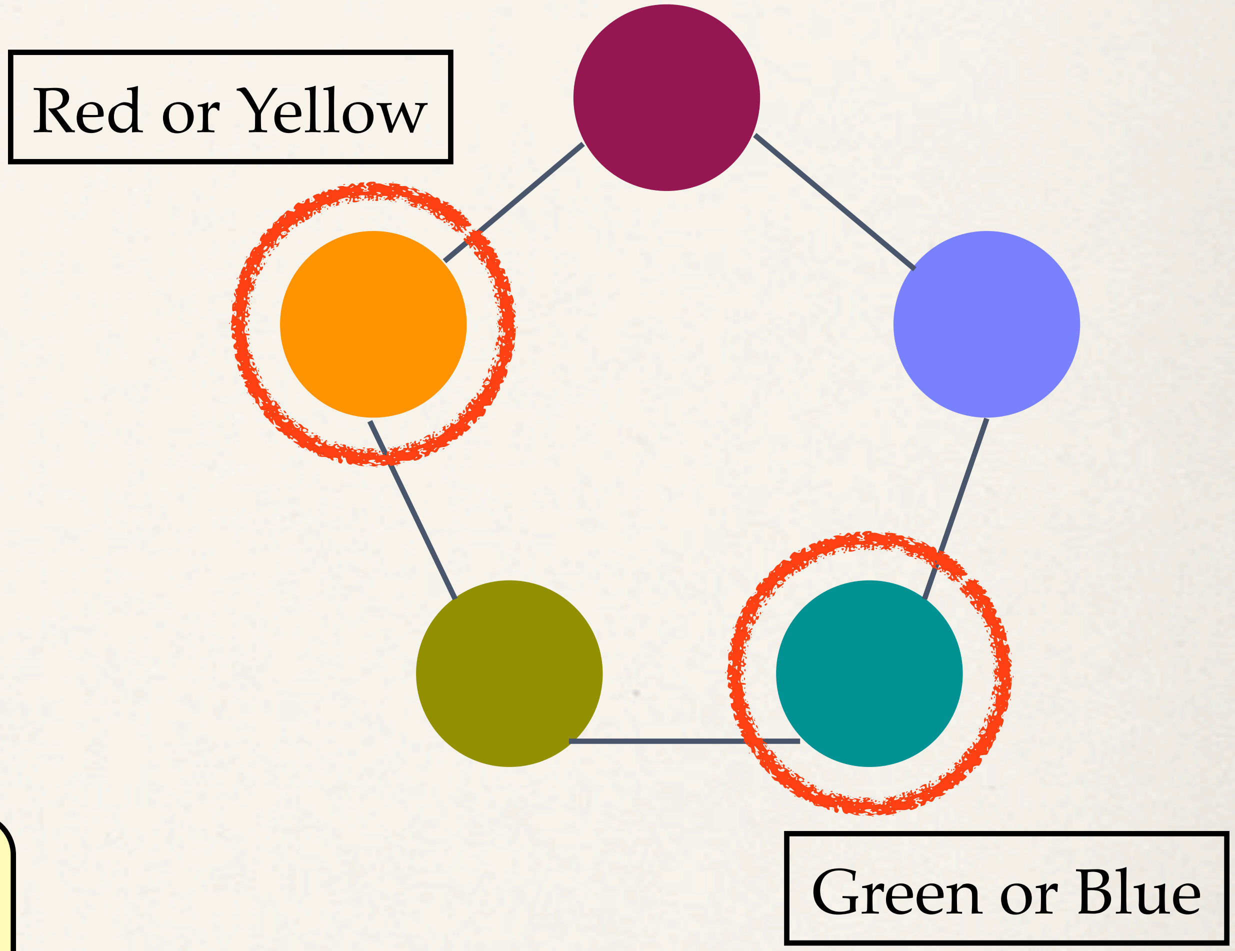


***α(G)***
THE INDEPENDENCE NUMBER

CONFUSABILITY GRAPH

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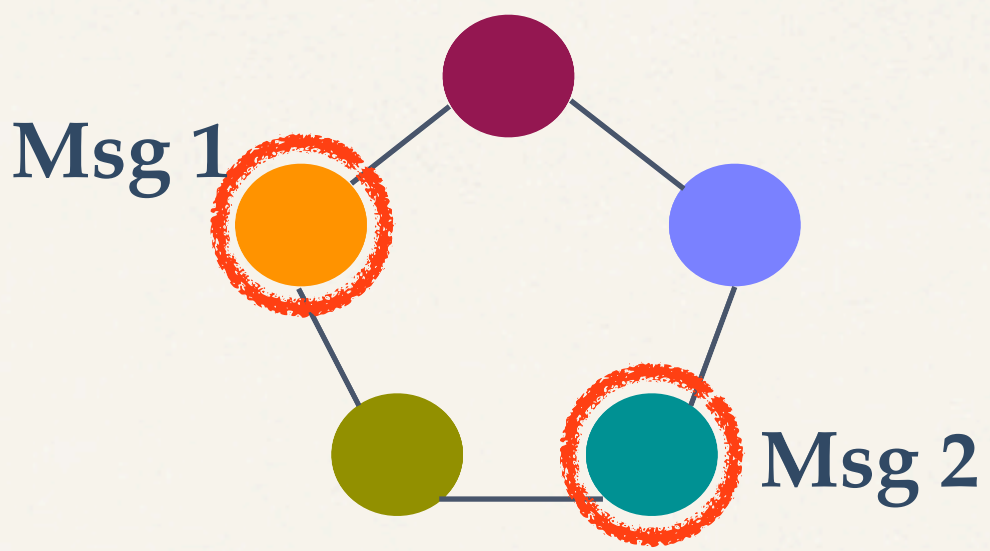

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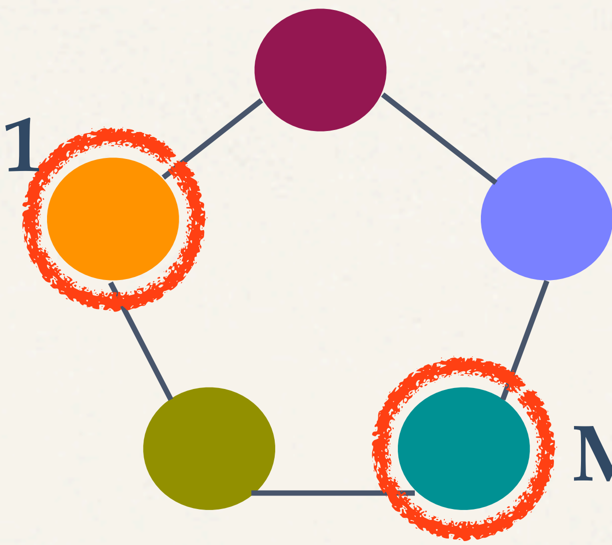
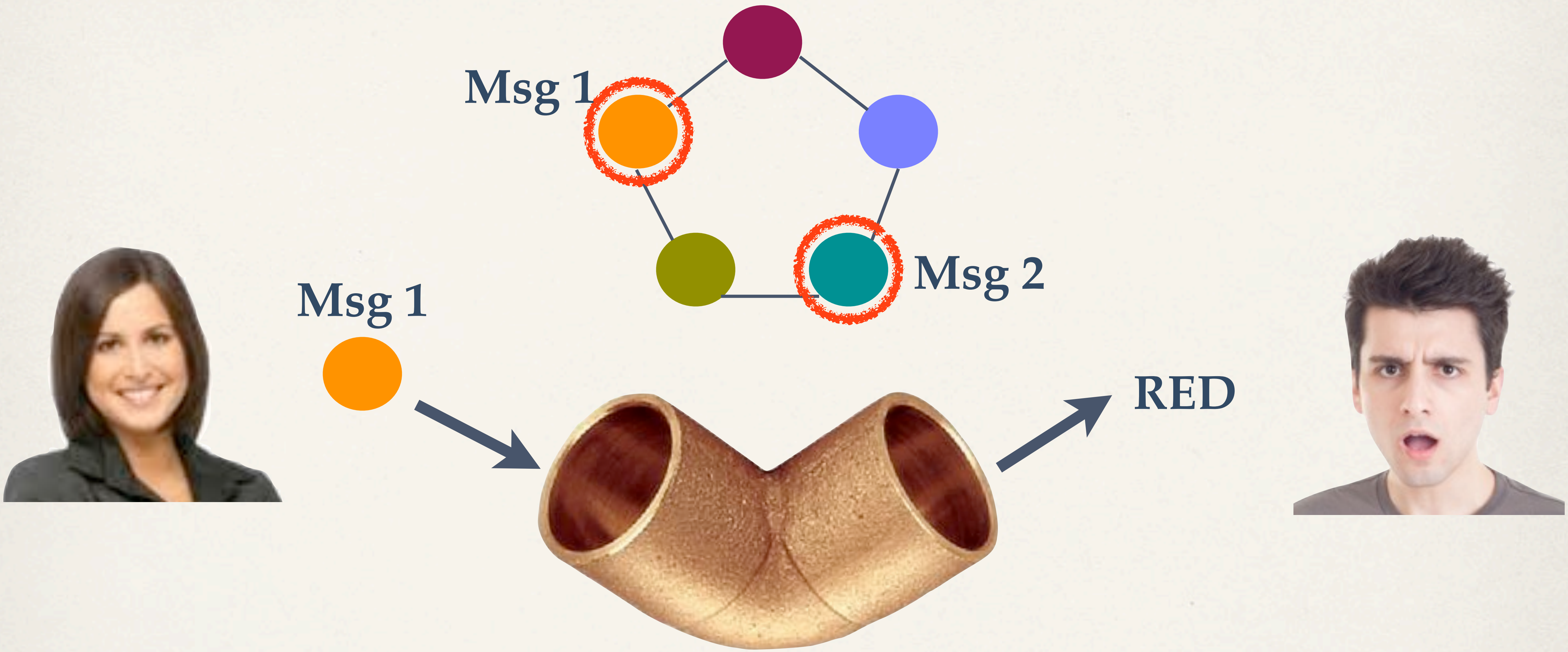



Msg 1

# Zero-Error Communication



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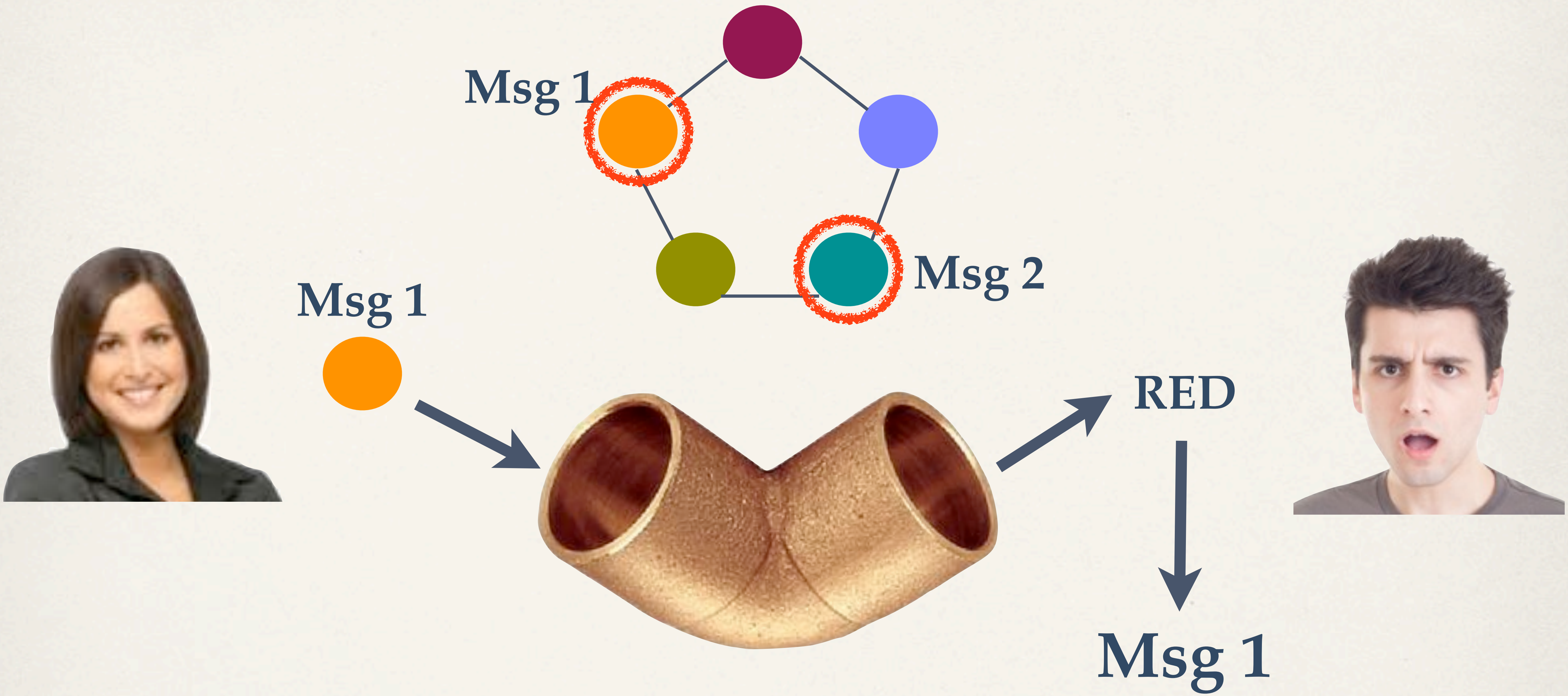

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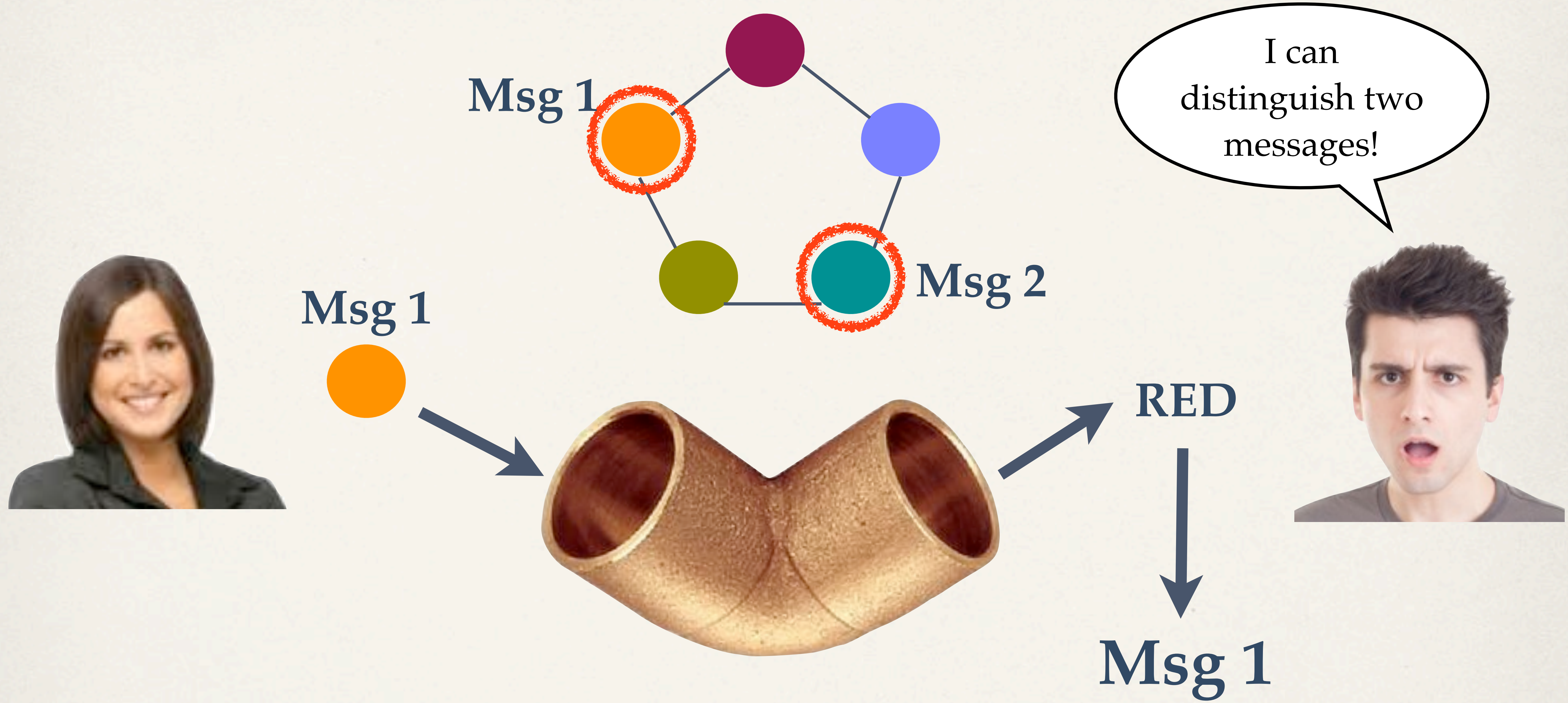

# Improving communication with entanglement

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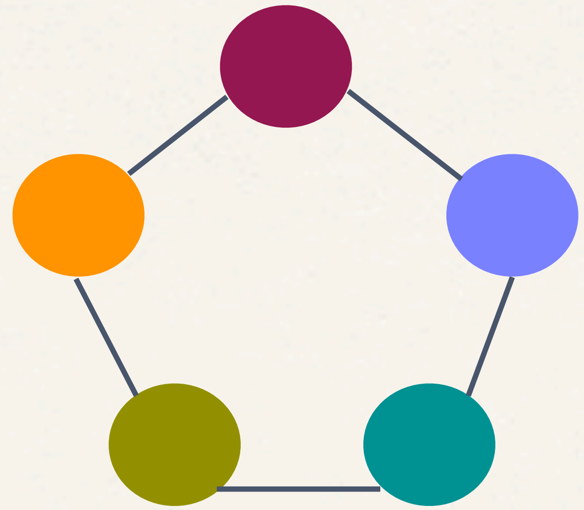

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**Msg *i***
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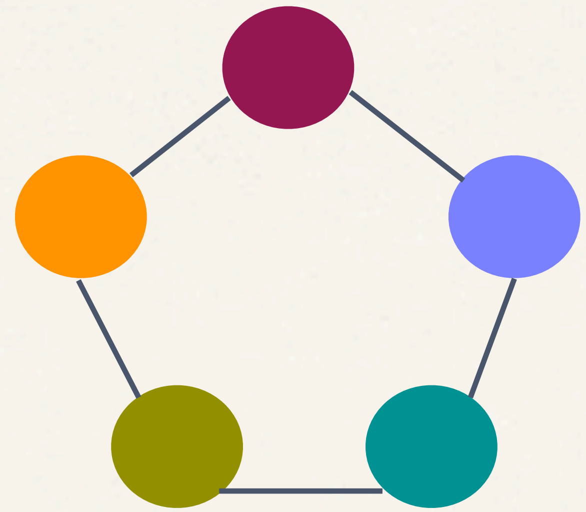

$|\psi\rangle$

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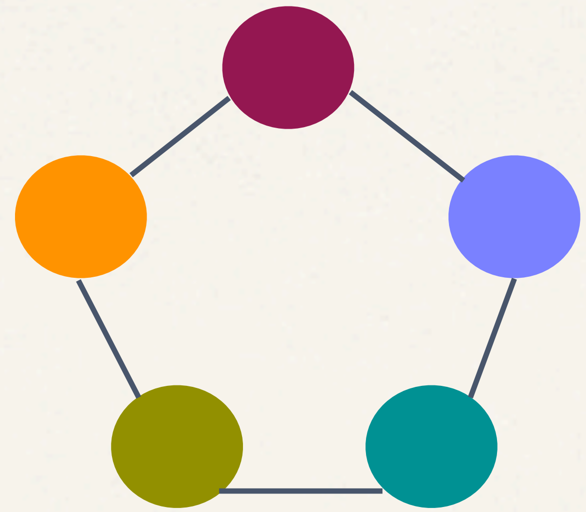

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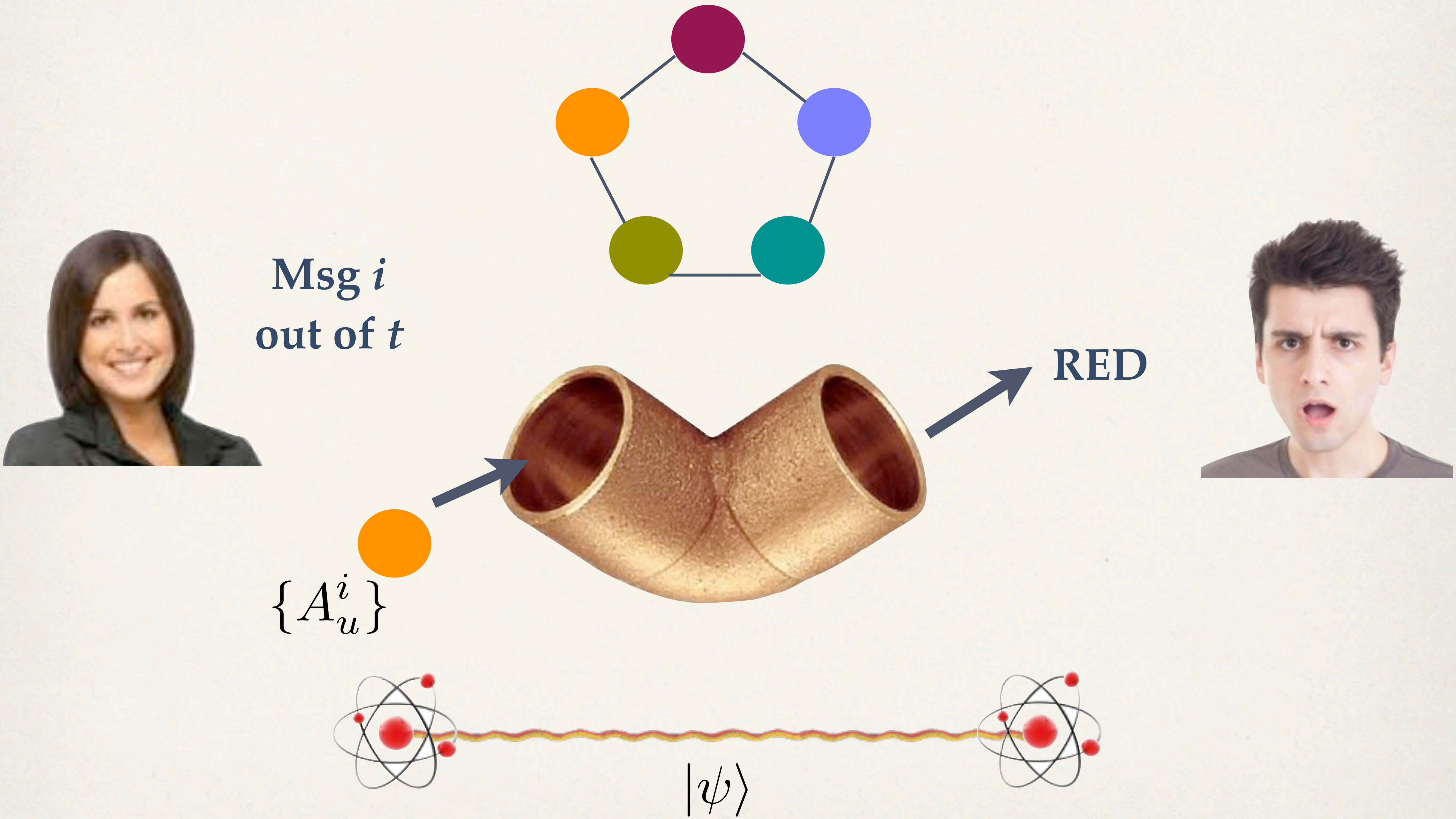

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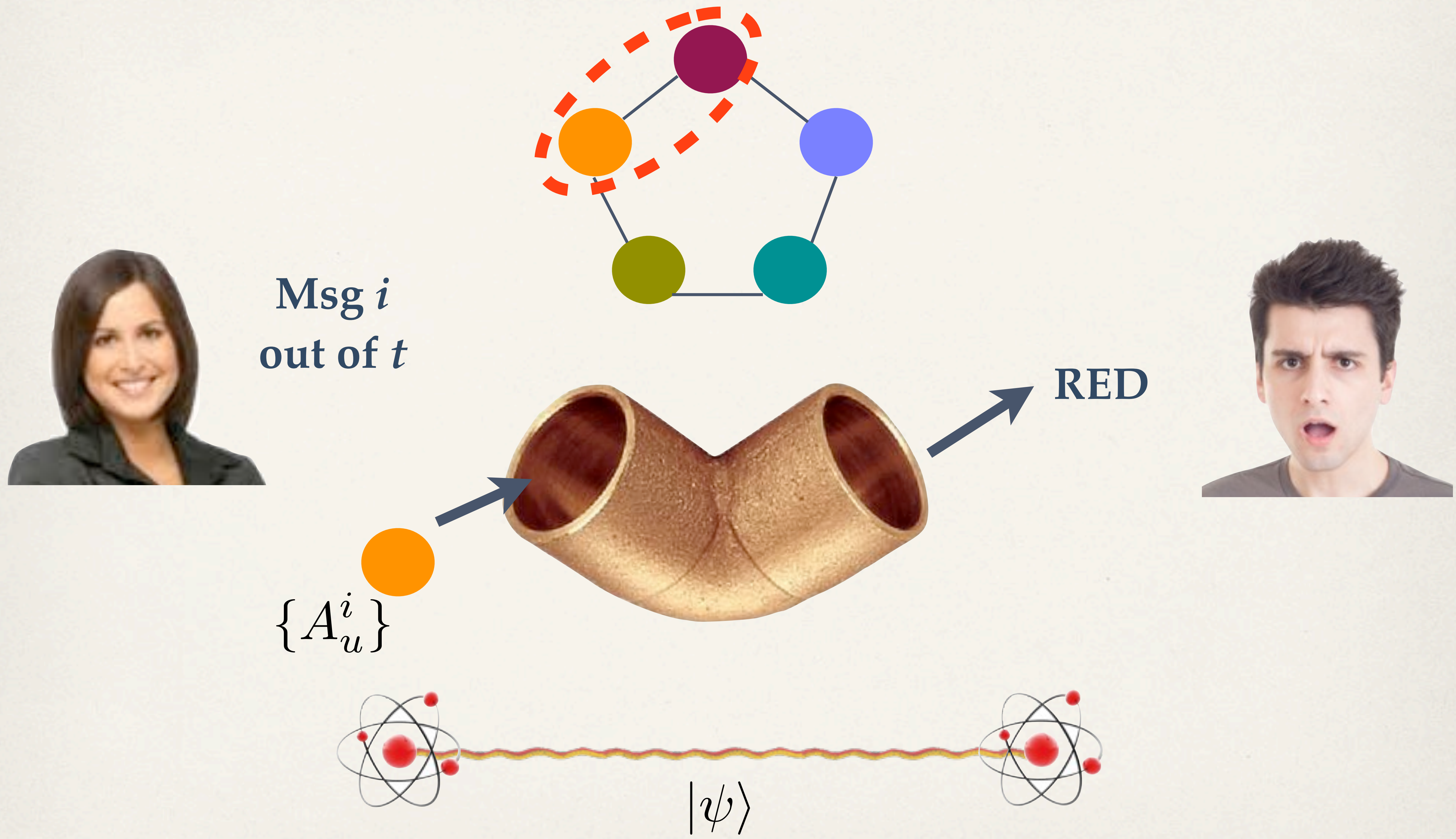

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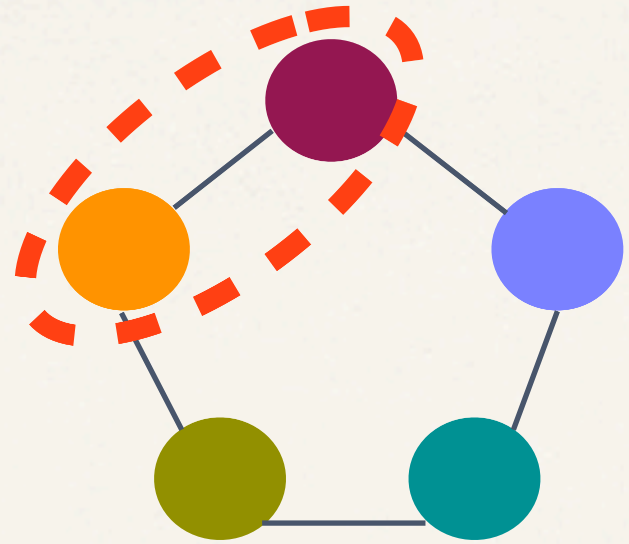

Msg $i$
out of $t$

RED

$\{A_u^i\}$

$|\psi\rangle$

$i \neq j$ and $a \simeq b$
$\Rightarrow$ *Orthogonal subspaces*

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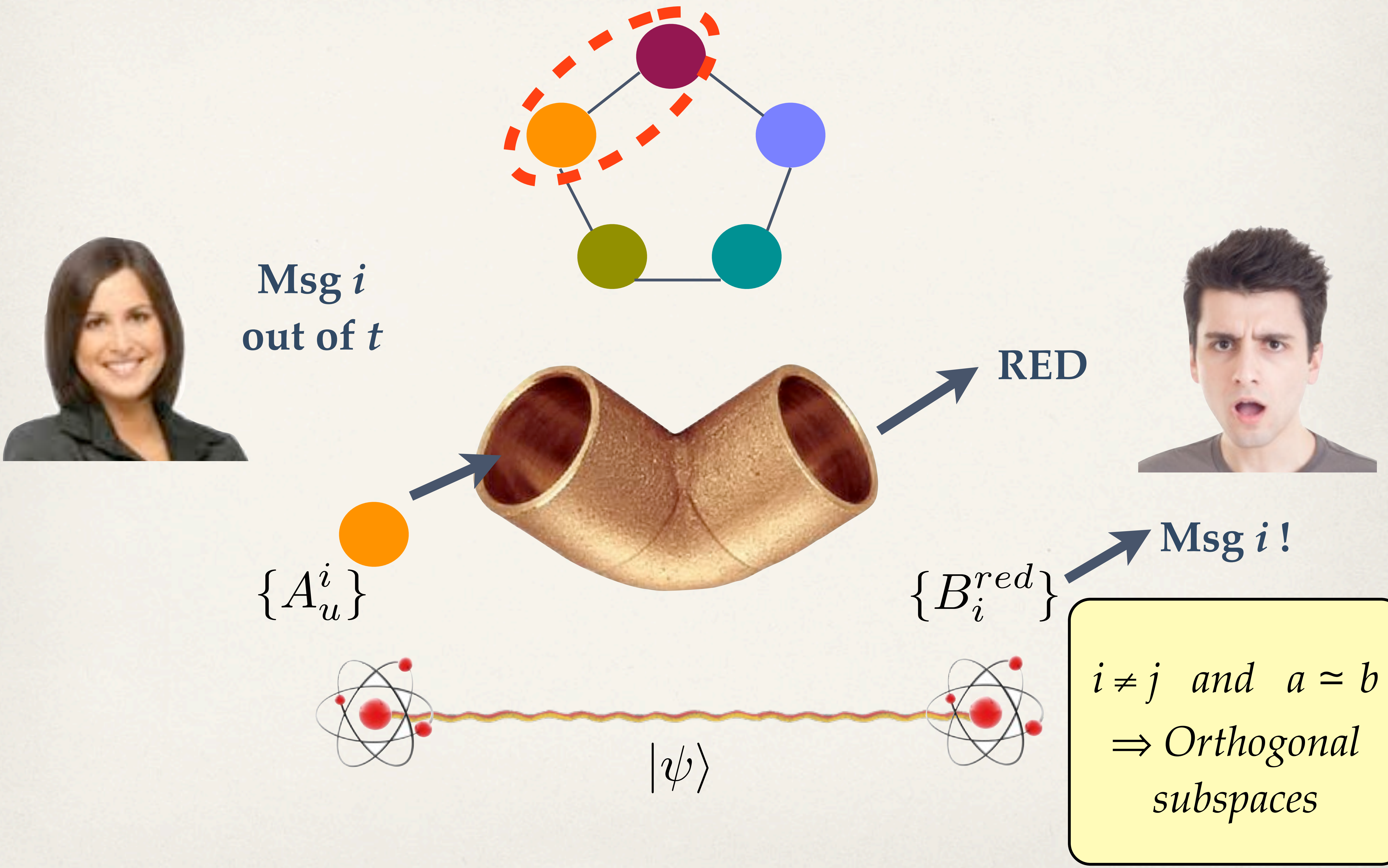

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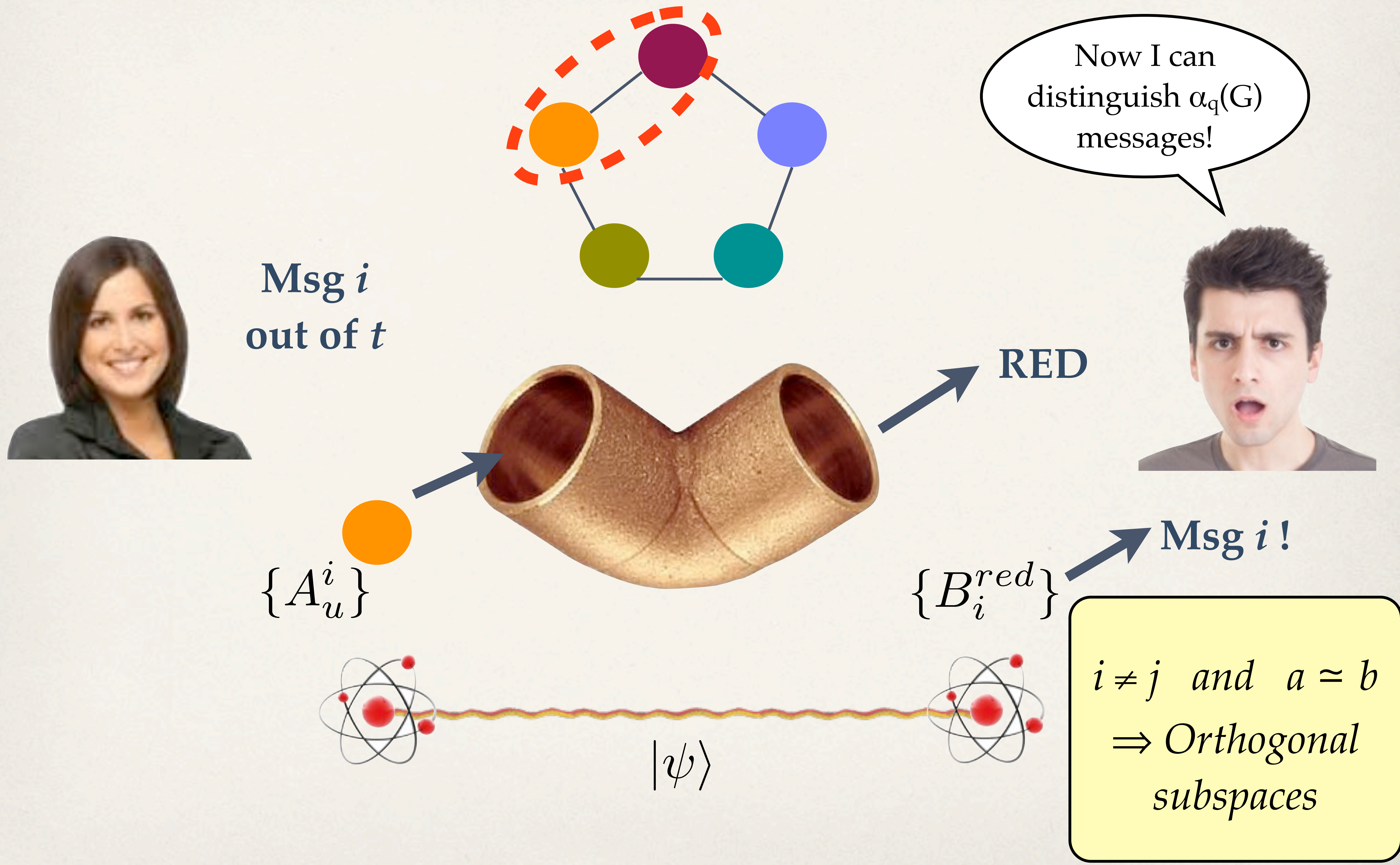

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- **Are there graphs for which $\alpha_q(G) < \alpha^*(G)$?**
- Also: ***Computability? Dimension of entangled state?***

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# Generic nonlocal game

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- Non-local game:

$x \in X$

$y \in Y$

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Probability distribution on inputs $\pi(x, y)$

Verification (payoff) function $\lambda : X \times Y \times A \times B \to \mathbb{R}$

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  - $\omega^*(\mathcal{G}) = \sup_{\psi\{A_a^x\}\{B_b^y\}} \sum_{xyab} \pi(x,y)\lambda(x,y,a,b)\langle\psi|A_a^x \otimes B_b^y|\psi\rangle$

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- We import tools from our quantum graph parameters framework

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  - Slightly different, topological approach
- Here, we heavily build-up from CSW and focus on non-local games.
  - We ***lower bound*** the quantum value with the *quantum independence number.*

# The game graph

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- We associate a graph $G$ as follows:

$$V(G) = \{xyab : \lambda(xyab) = 1\}$$

$$E(G) = \{\{xyab, x'y'a'b'\} : (x = x' \wedge a \neq a') \vee (y = y' \wedge b \neq b')\}$$

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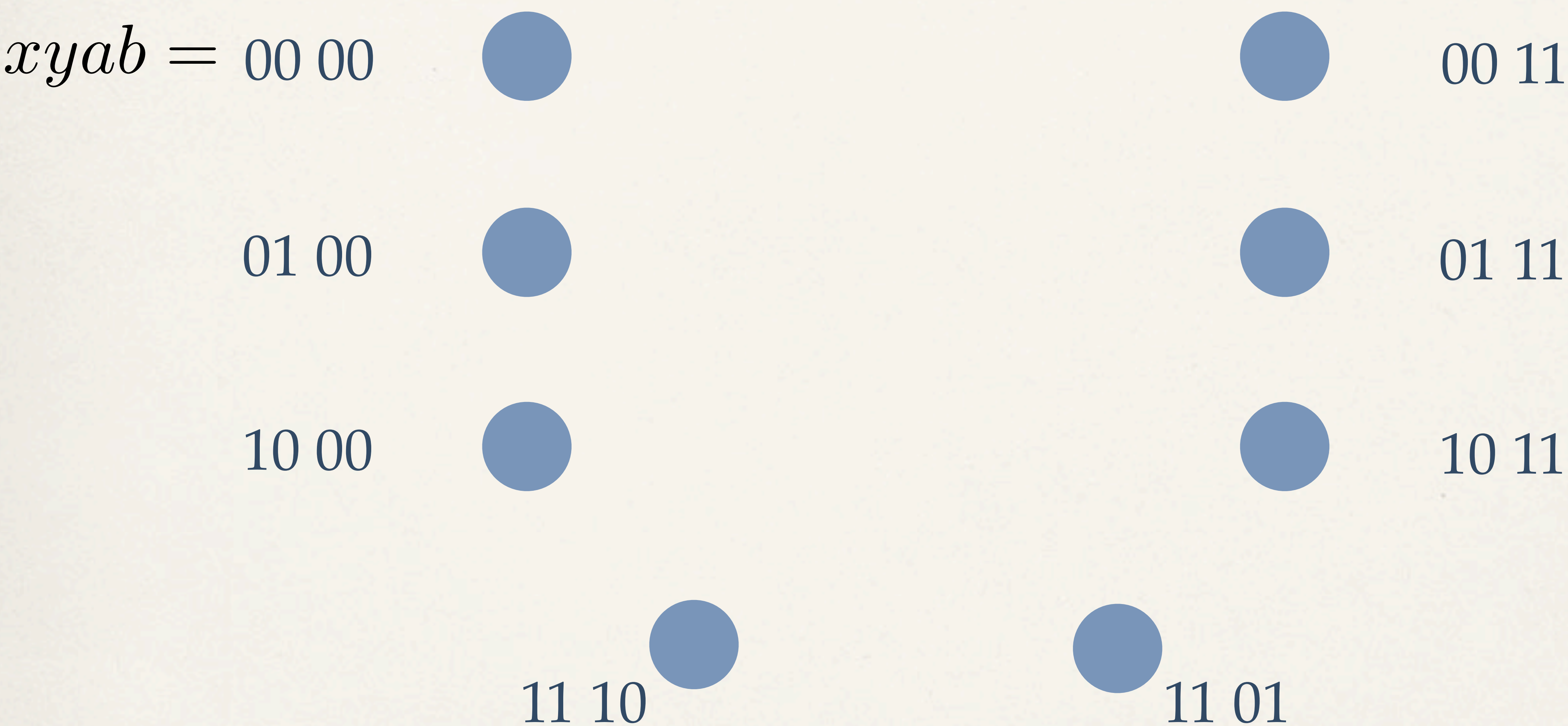
- A simple example is given by CHSH.

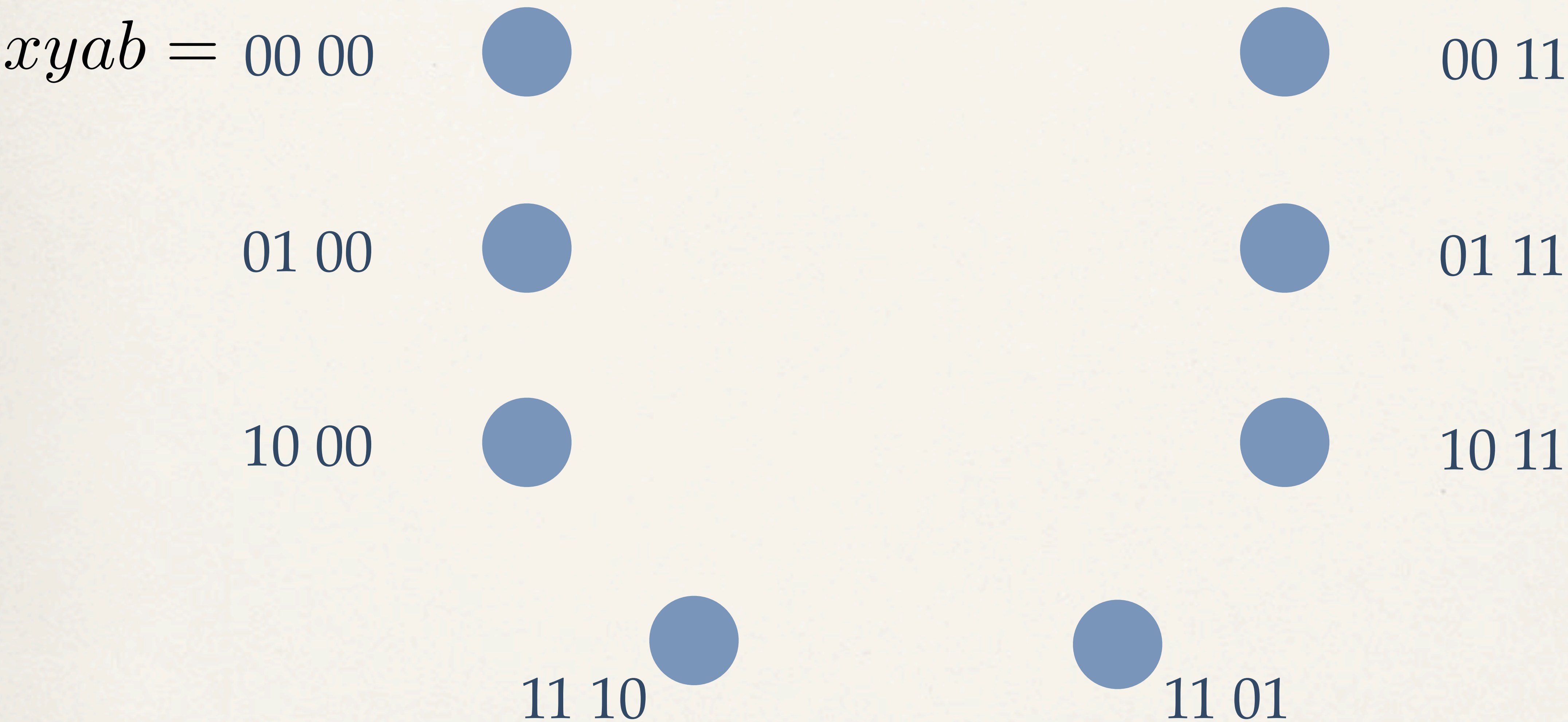
# CHSH Game Graph

$$\lambda(xyab) = 1 \iff a \oplus b = x \cdot y$$



# CHSH Game Graph

$$\text{edge if } (x = x' \wedge a \neq a') \vee (y = y' \wedge b \neq b')$$



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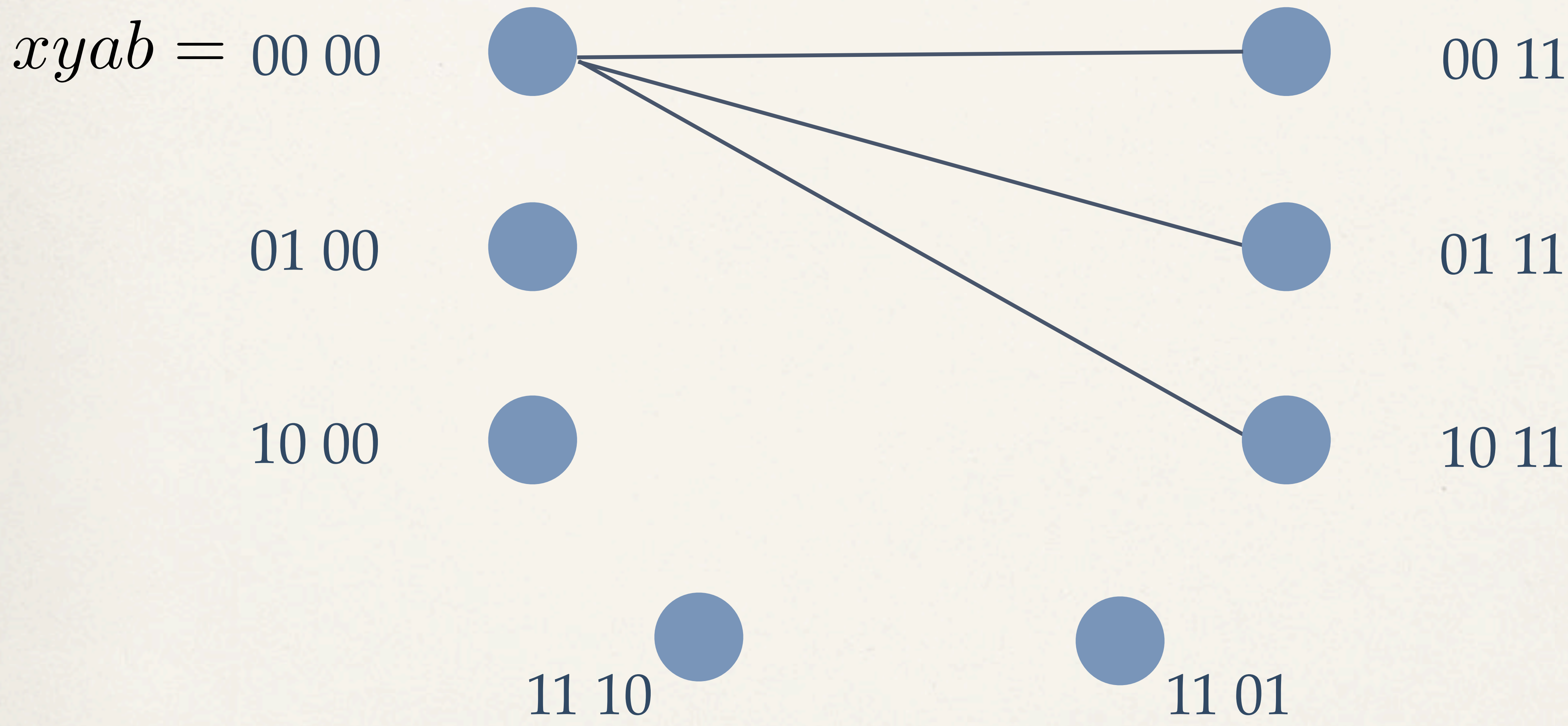

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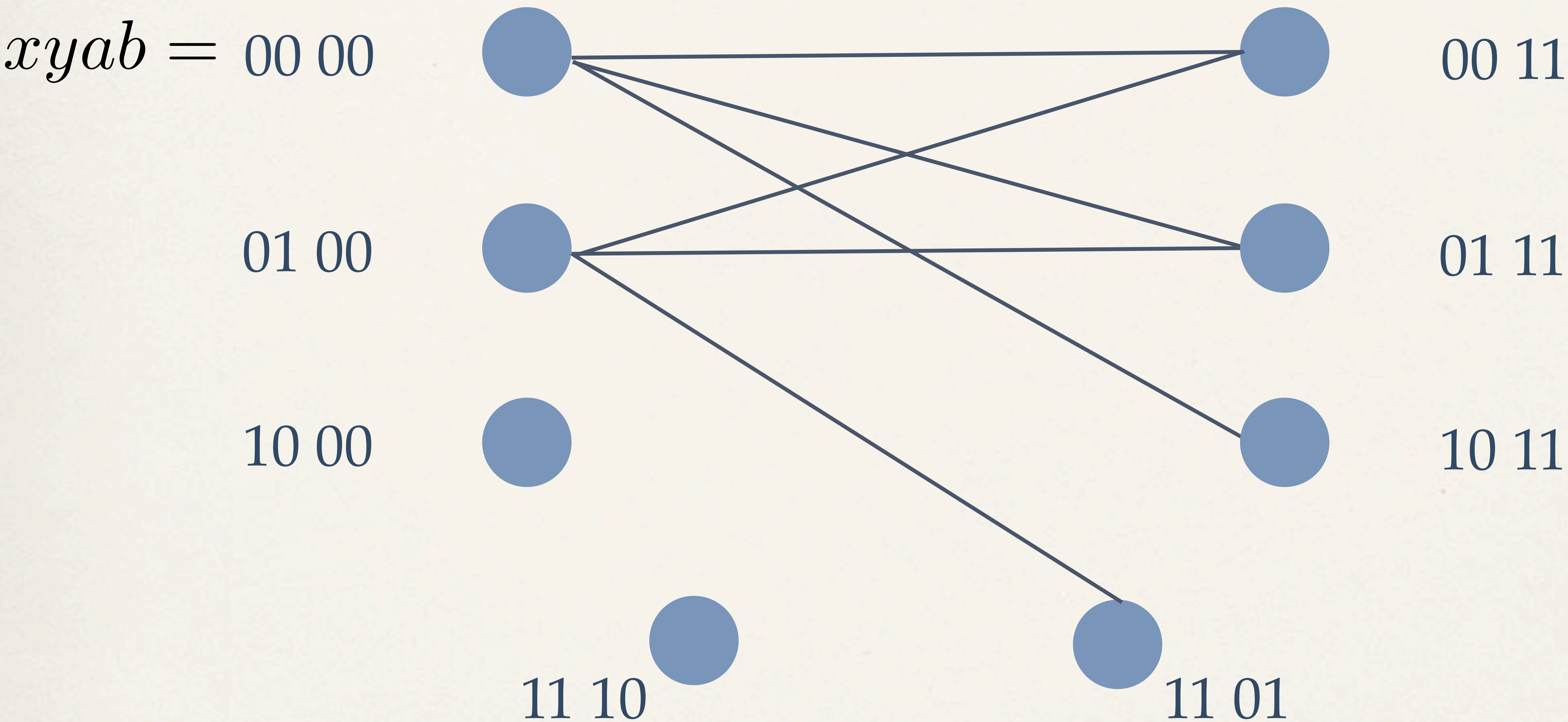

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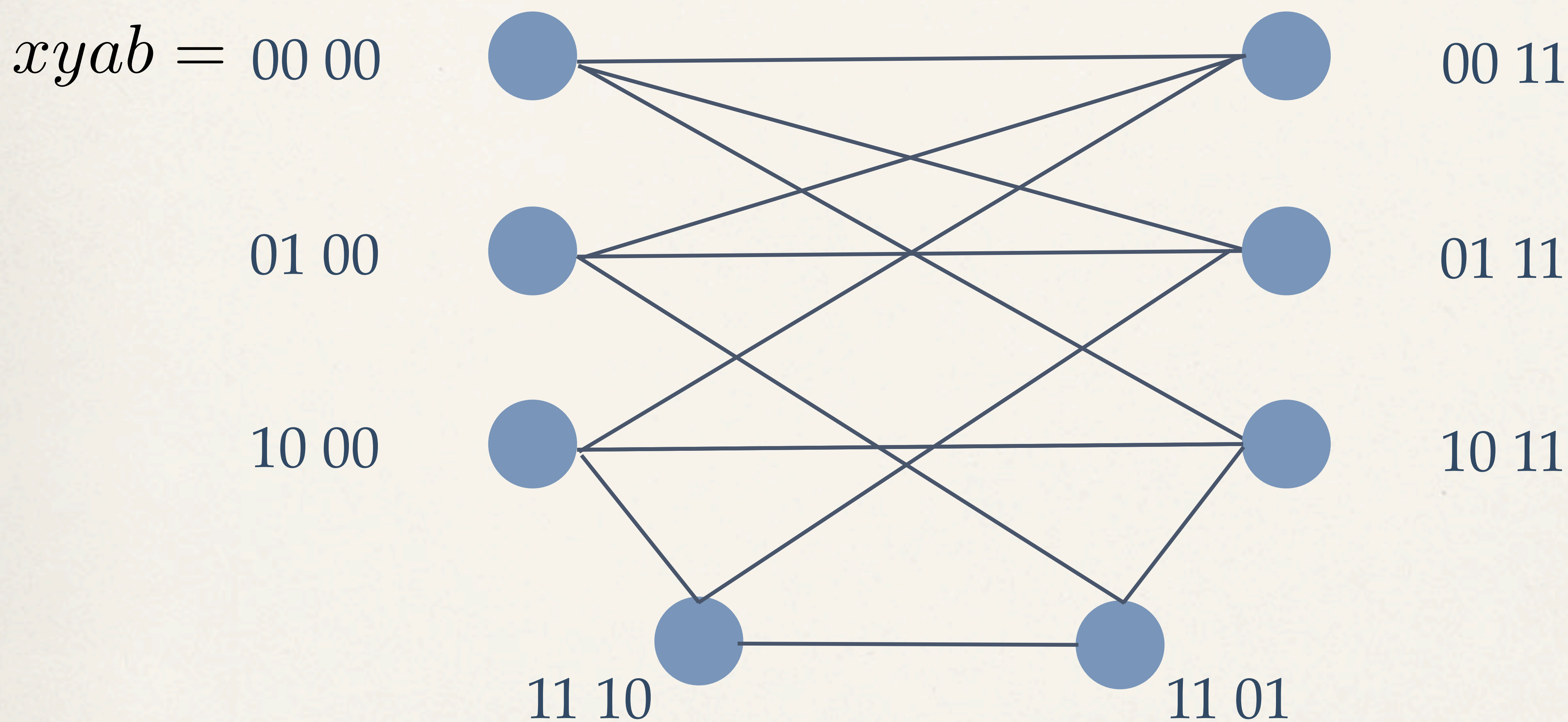

# Classical value

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- Let us see an intuition

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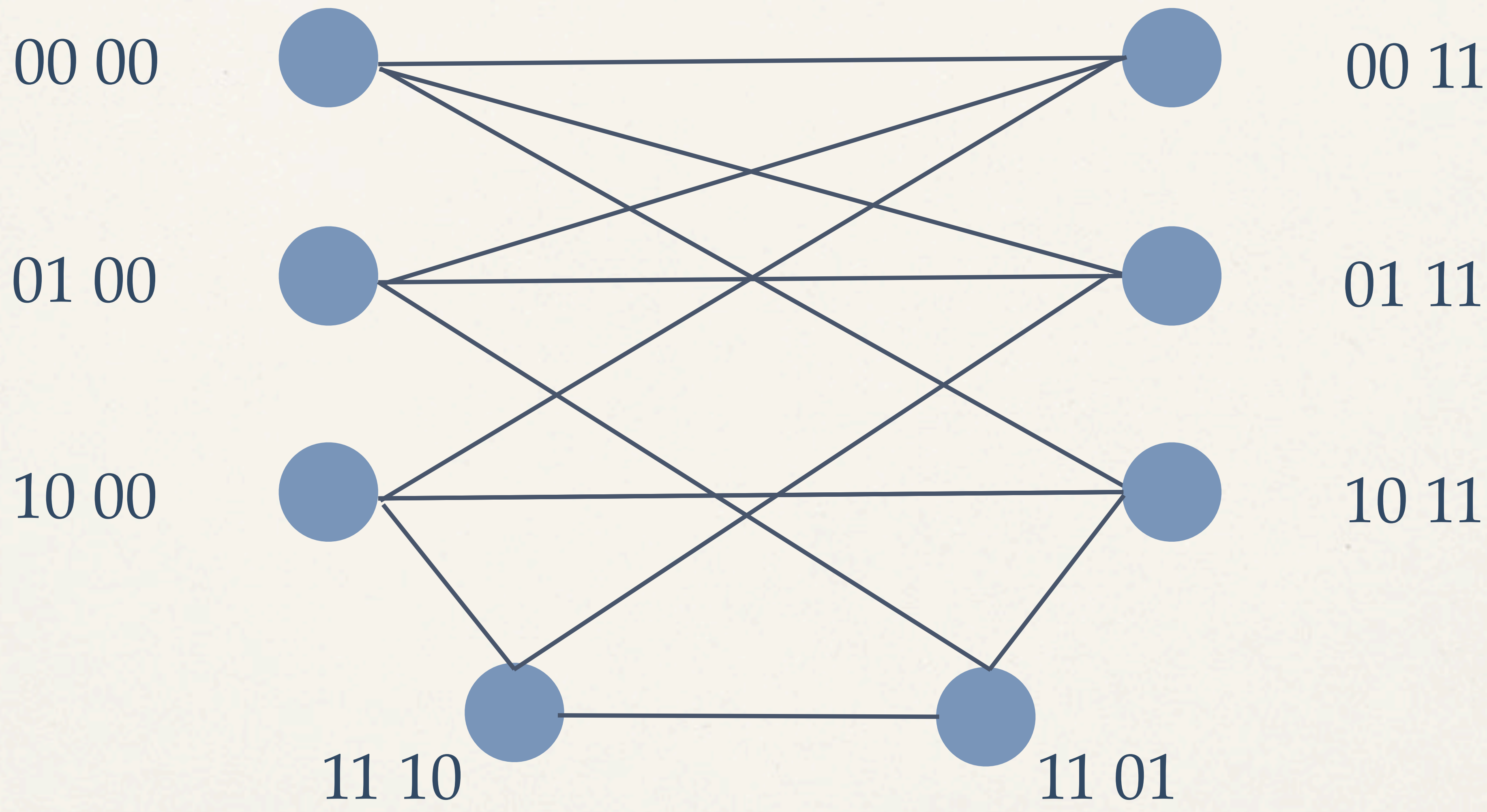

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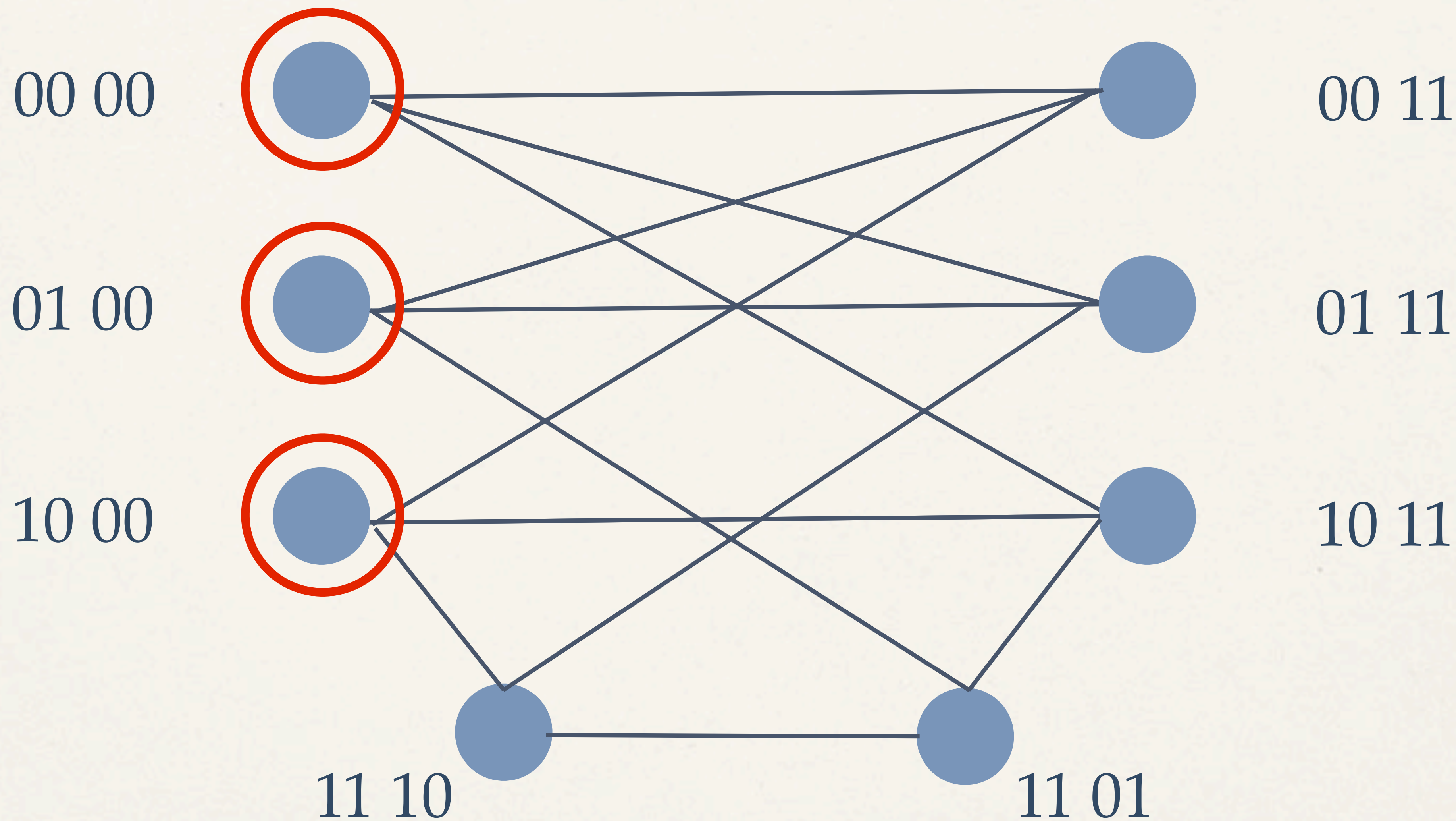

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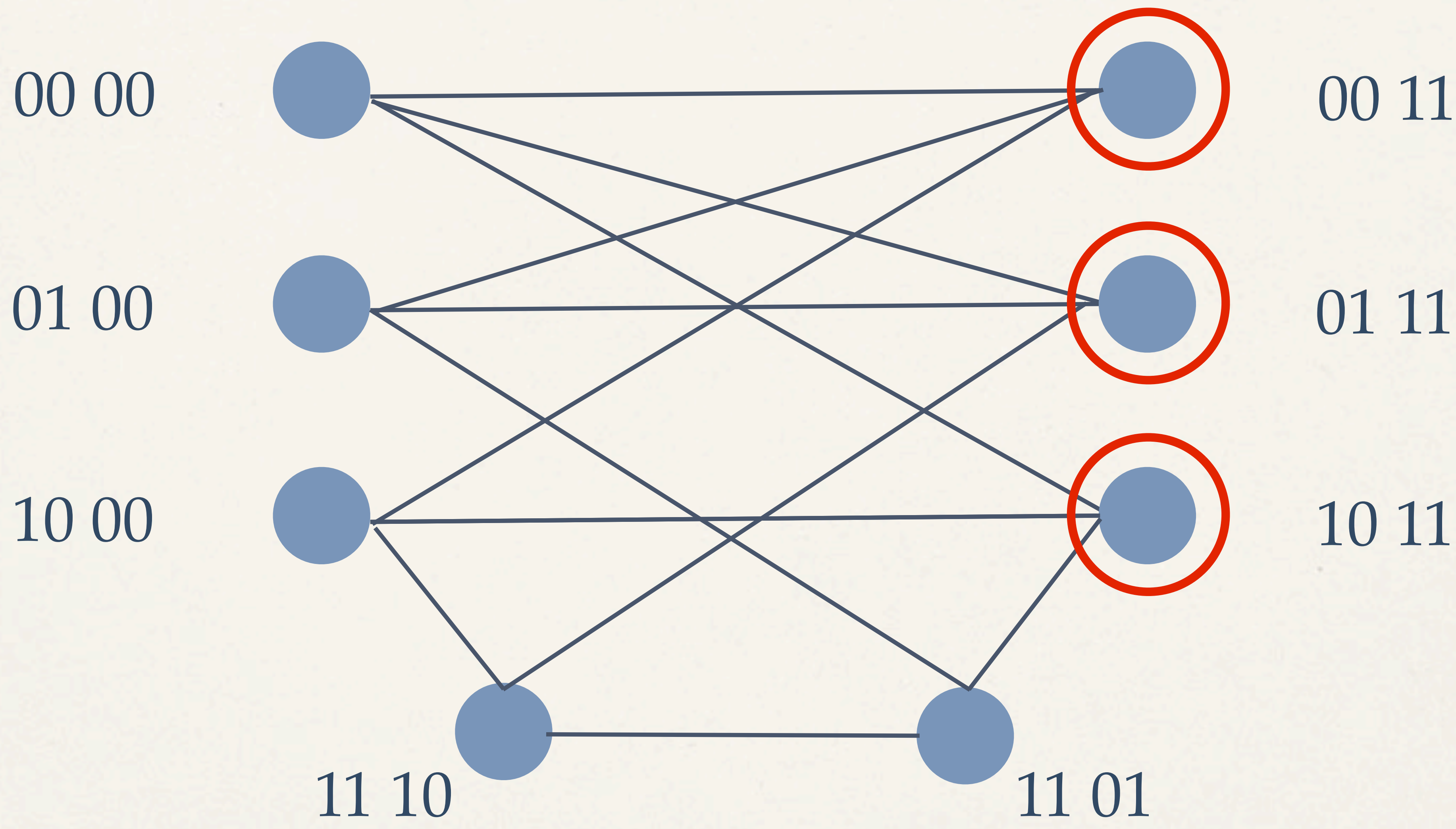

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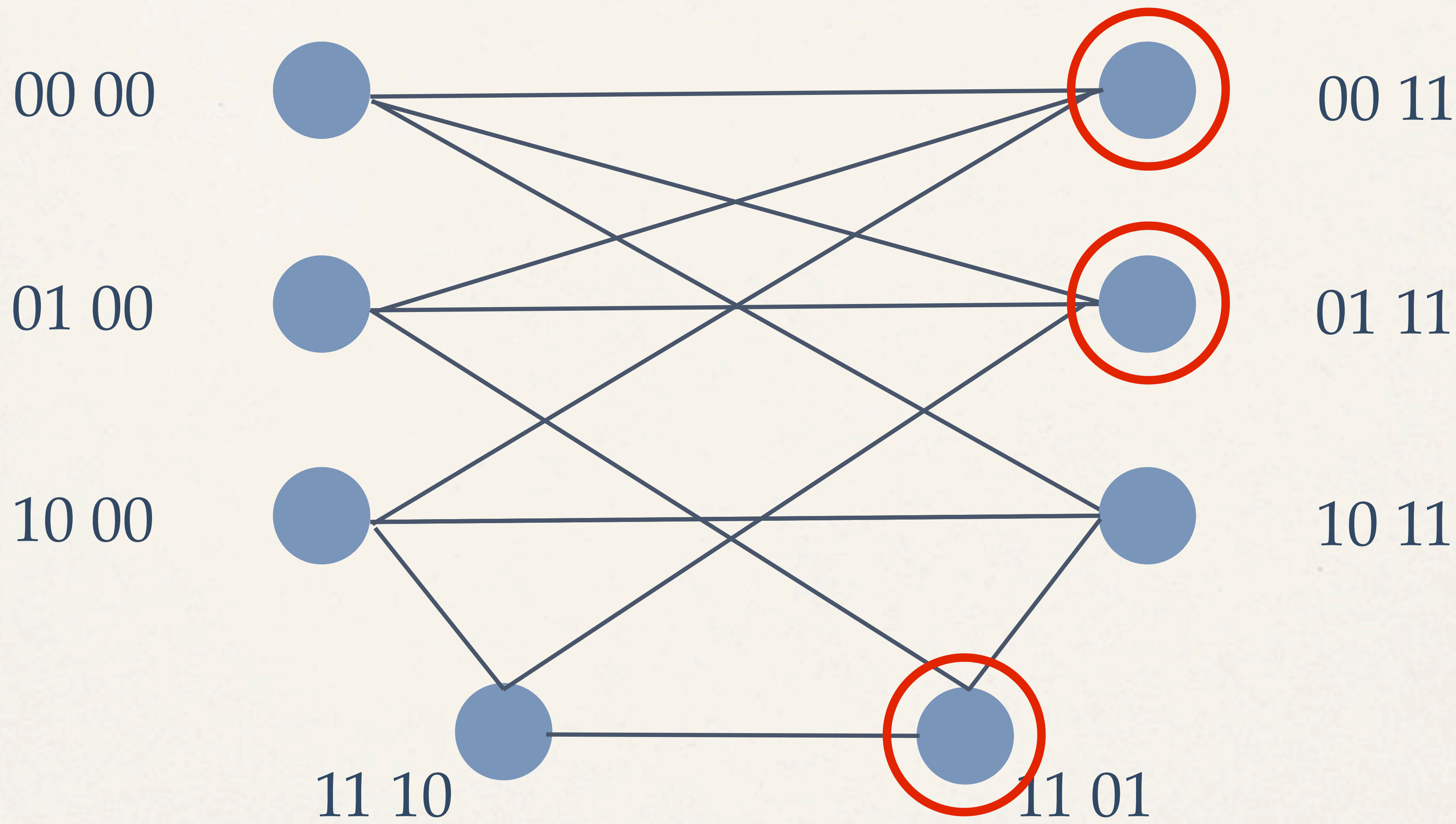

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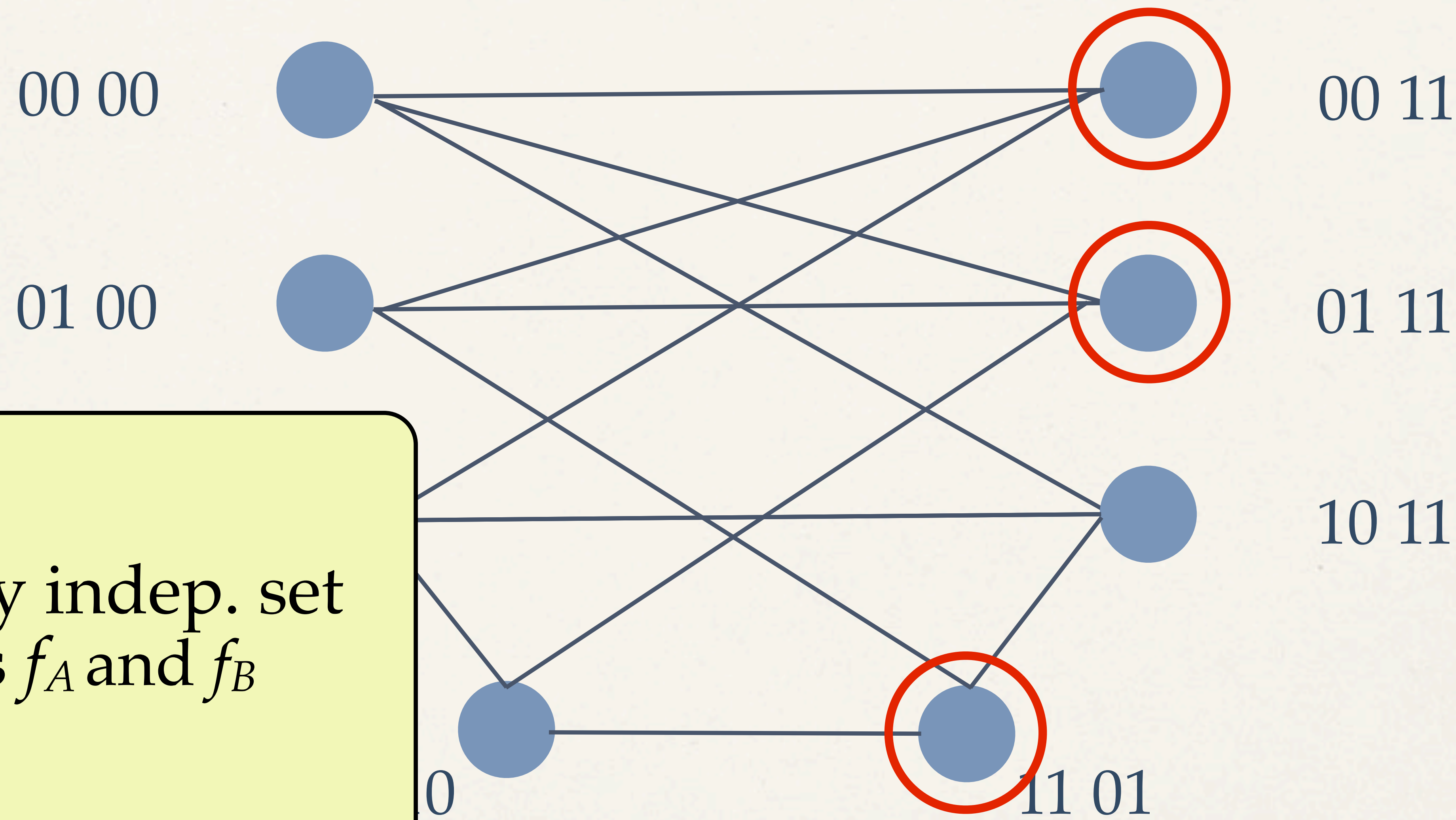


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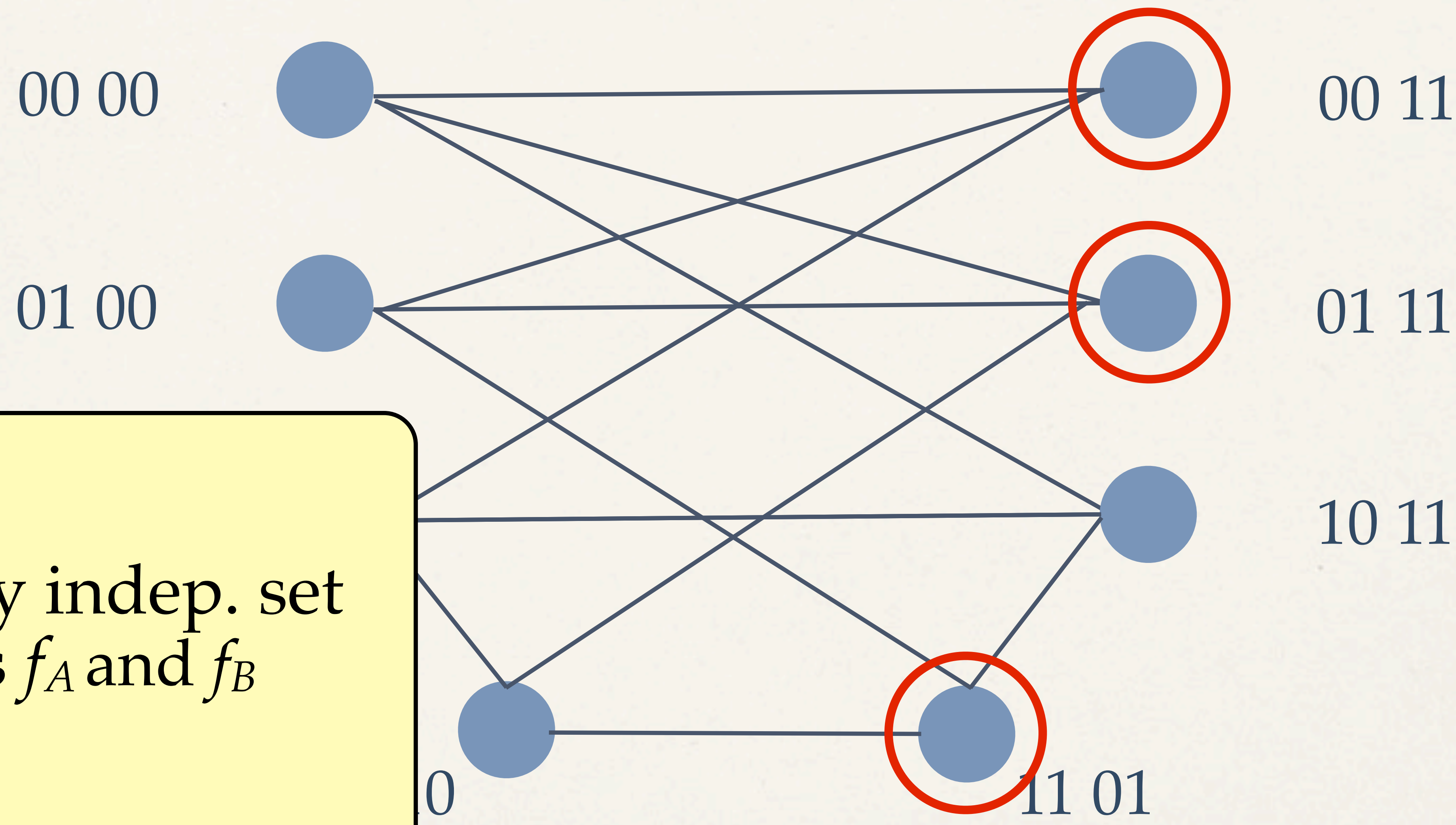


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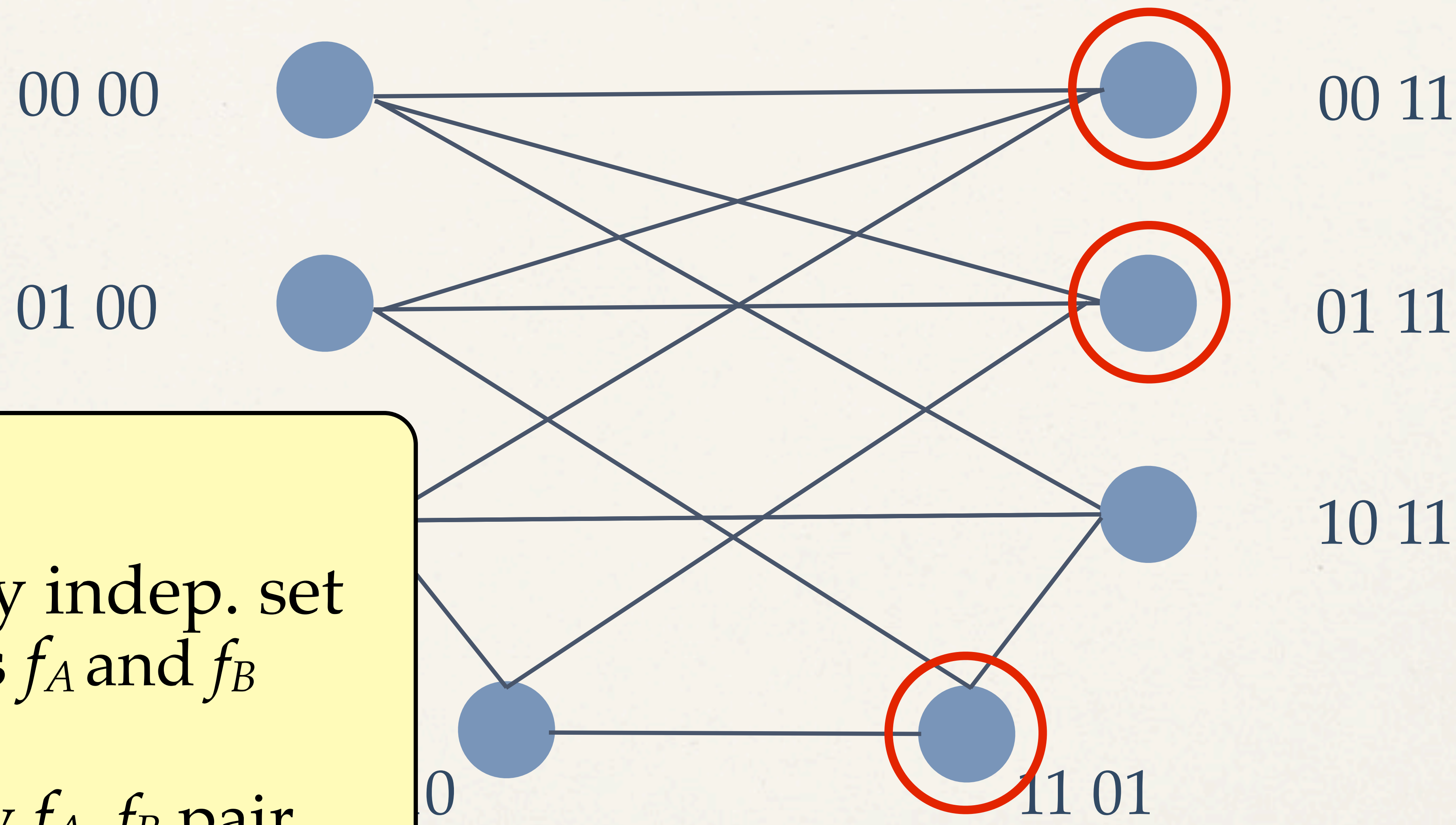


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- Every $f_A$, $f_B$ pair give an indep. set.

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# Lower bound proof sketch

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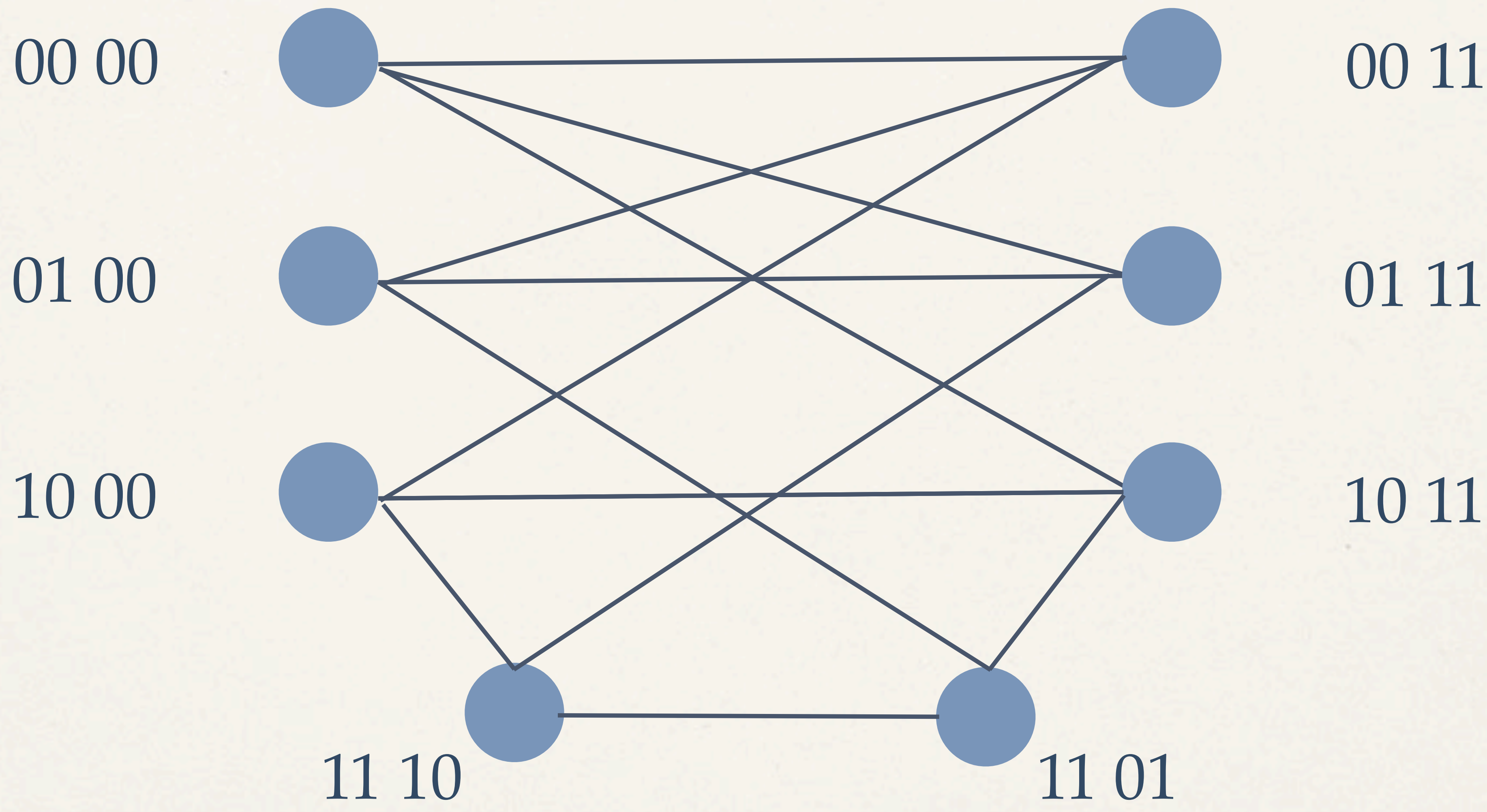

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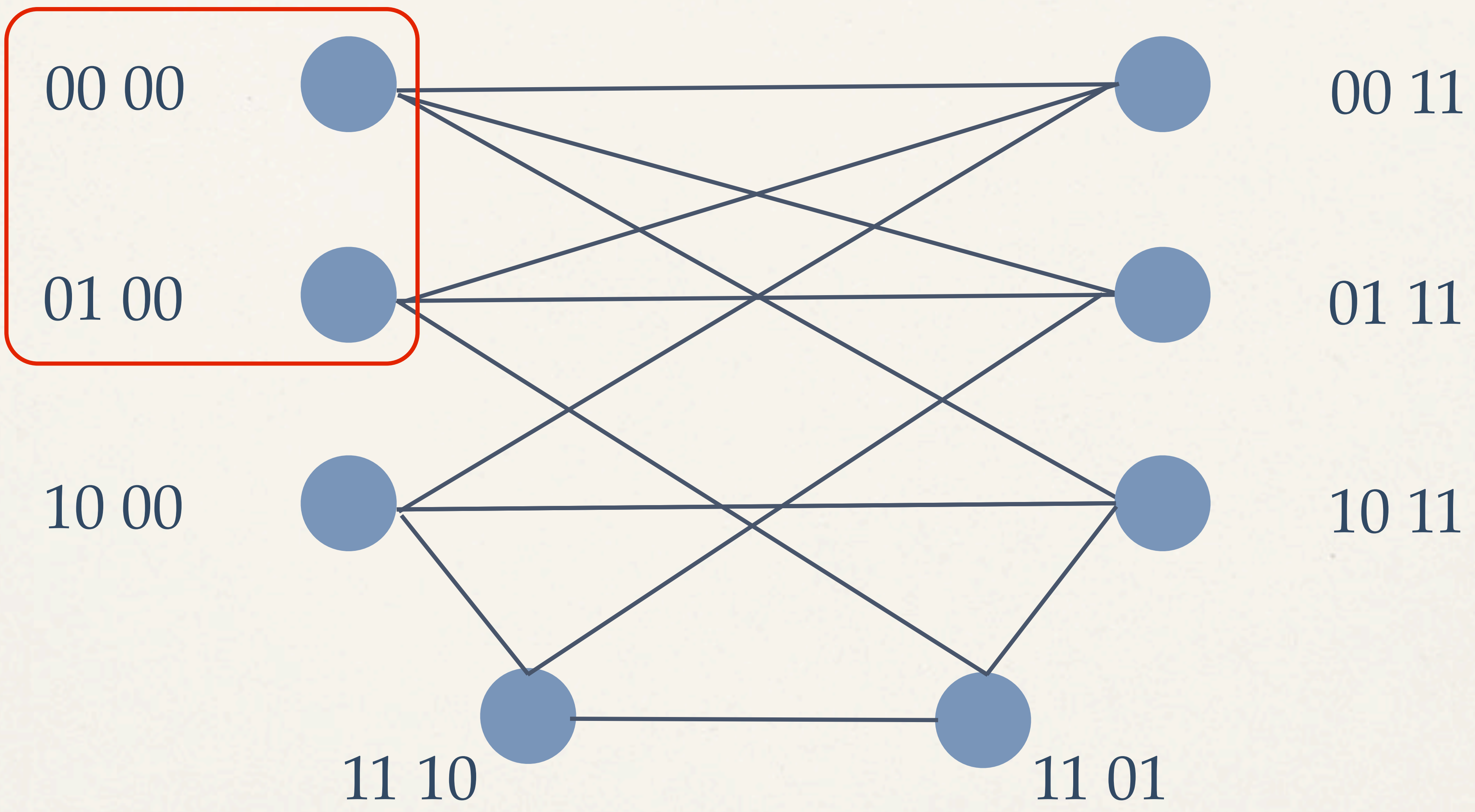
$P_0^0$



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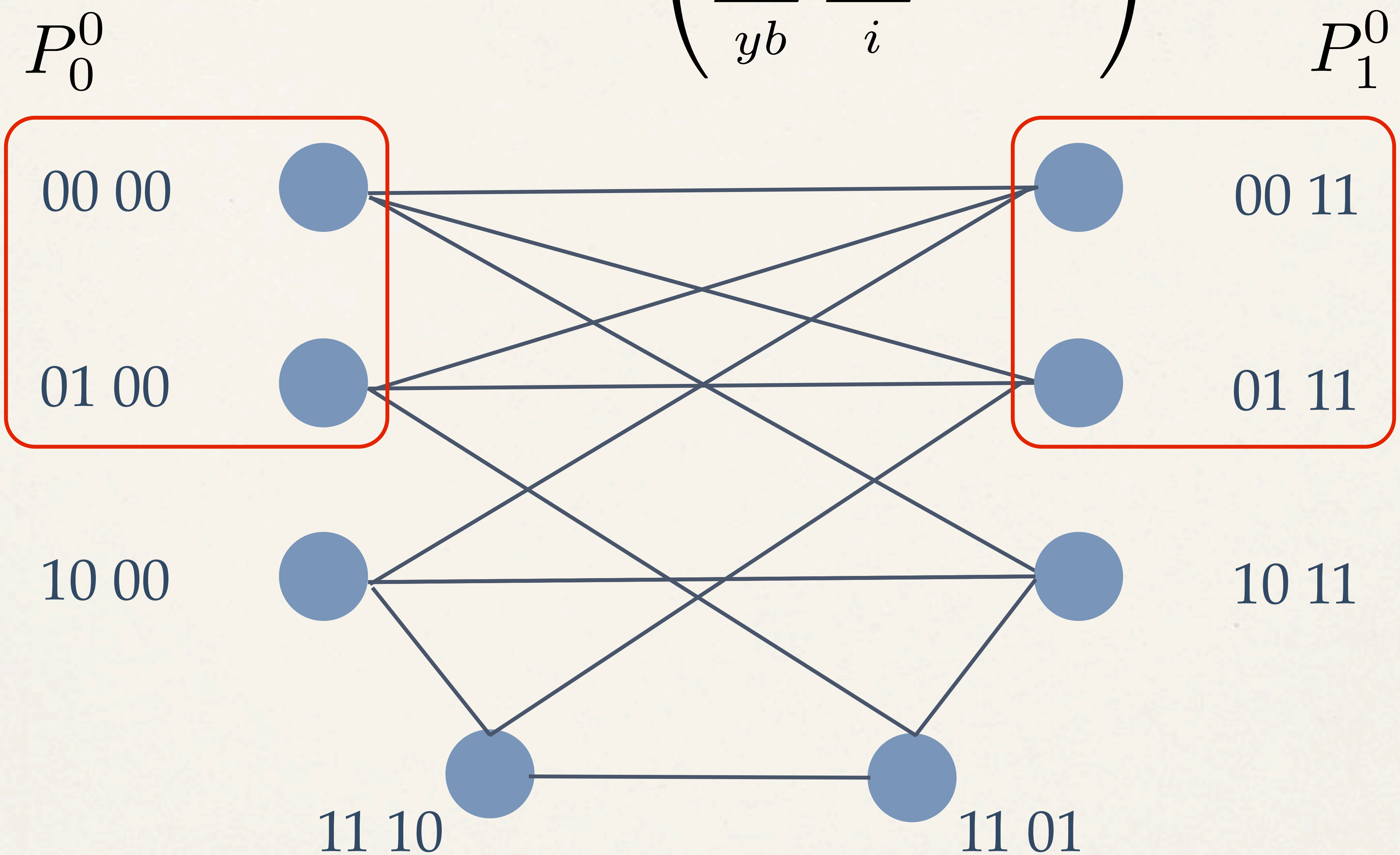

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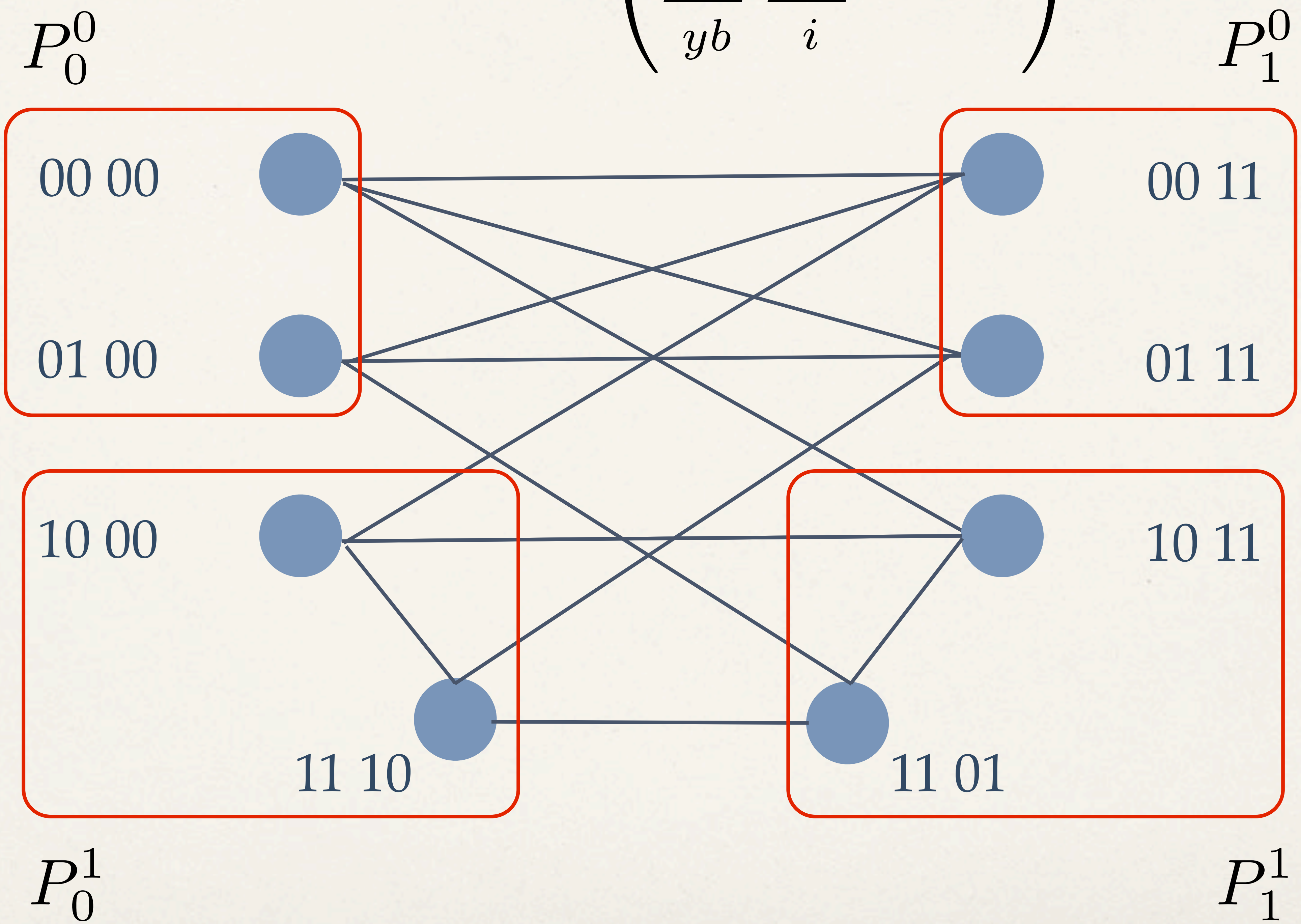

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$$\alpha_q$$

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- They are complete projective measurements
- We calculate:

$$\omega^*(\mathcal{G}) \geq \frac{1}{|X \times Y|}\sum_{xyab}\langle\psi|P^x_a \otimes P^y_b|\psi\rangle \geq \cdots \geq \frac{\alpha_q(G)}{|X \times Y|}$$

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- All the results we have shown generalize to any input distribution and any $\lambda : X \times Y \times A \times B \to \mathbb{R}$ by considering *weighted graphs*
- We define a quantum independence number for weighted graphs (not a homomorphism game!)
- We have a class of nonlocal games for which our lower bound is tight

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- We can “reverse engineer” graphs in order to get large Bell violations and large separation in channel capacity
- Graph homomorphism games more powerful? [Mančinska and Roberson ’13]