Strong trapping in discrete time quantum walks

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with
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PRA **91**, 022308 (2015)

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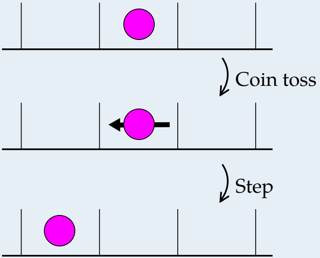
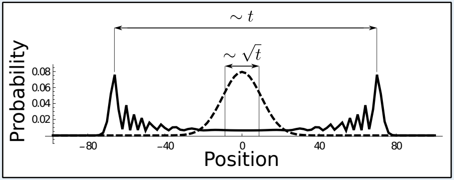
Classical walks vs. Quantum walks

Classical walks

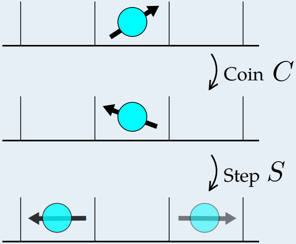


- Diffusive spreading



Discrete time quantum walks

- Nontrivial extension
  Y. Aharonov et al. (1993),
  D. A. Meyer (1996)
- Time evolution: $U = S \cdot (I_P \otimes C)$
  $S|x\rangle_P \otimes |\rightarrow\rangle = |x+1\rangle_P \otimes |\rightarrow\rangle$
  $S|x\rangle_P \otimes |\leftarrow\rangle = |x-1\rangle_P \otimes |\leftarrow\rangle$
  $C \in SU(2)\,, \quad I_P = \sum_x |x\rangle_P \langle x|_P$



- Ballistic spreading

Trapping in homogeneous quantum walks?

- Spatial homogeneity: $U = S \cdot (I_P \otimes C)$
- Formal solution by the Fourier transformation:
  (position,coin) $\Leftrightarrow$ (momentum,coin)
  $\tilde{U}(k) = \begin{pmatrix} e^{-ik} & 0 \\ 0 & e^{ik} \end{pmatrix} \cdot C$

- Trapping: stationary (momentum independent) eigenvalue(s)
  $\Leftrightarrow$ flat bands in the quasi-energy [$w(k)$] spectrum
  $\Leftrightarrow$ $\det\left(\hat{U} - e^{i\omega(k)} I\right) = 0$, with $\frac{d\omega(k)}{k} = 0$
- Explicitly solvable: $\tilde{U}(k) = \begin{pmatrix} e^{-ik} & 0 \\ 0 & e^{ik} \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$
  $\omega(k,l)$ quasi-energies: $\cos\omega = e^{ik}\mathrm{Re}\,\alpha - i\alpha \sin k \Rightarrow \alpha \equiv 0$

- No "real" trapping in homogeneous 1D discrete time quantum walks
- 1D trapping with more internal coin states! (But also classically — Lazy walks)
  N. Konno et al. JPhysA **353**, 133 (2005)
  M. Štefaňák et al. PRA **90**, 012342 (2014)

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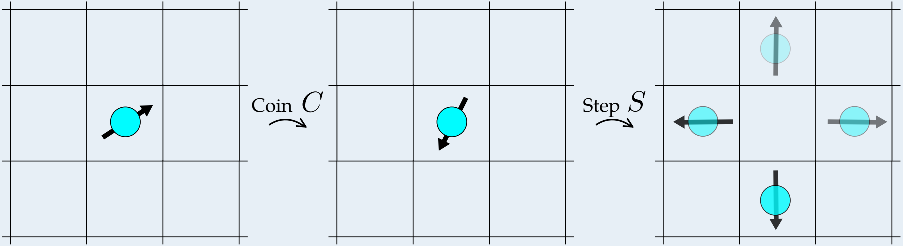
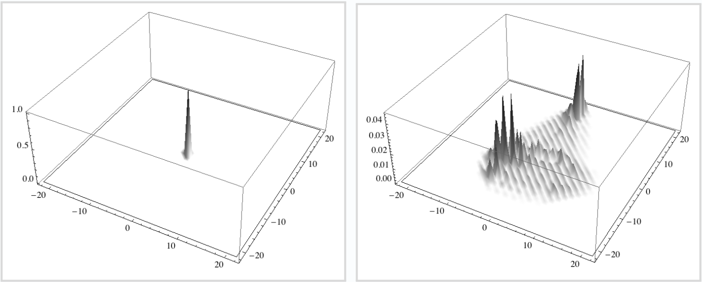
2D quantum walks — Trapping?

- Nearest neighbour steps, four dimensional coin space





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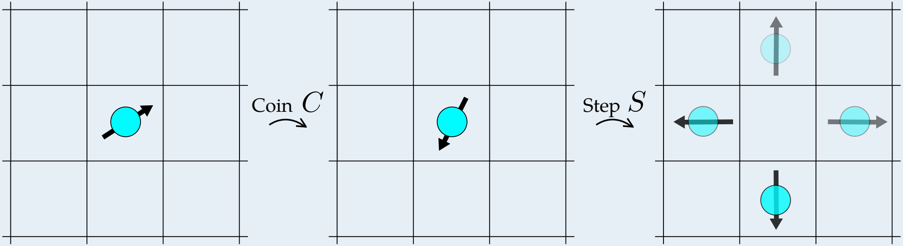

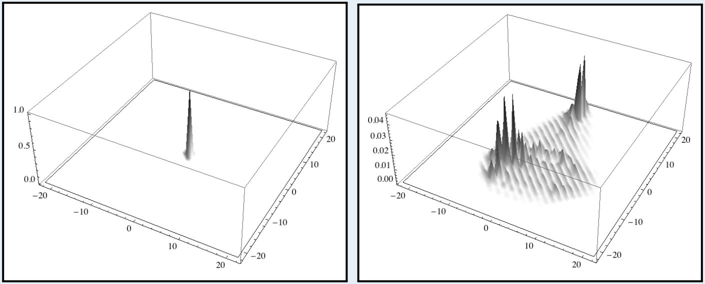

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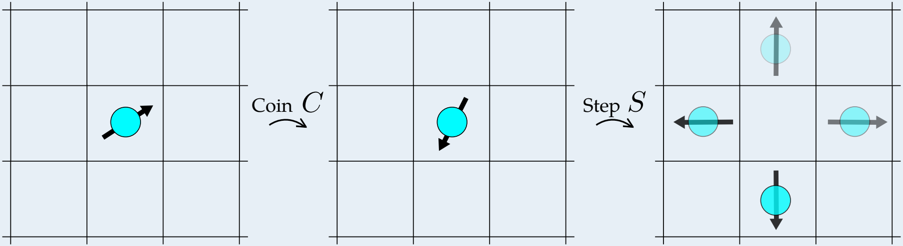


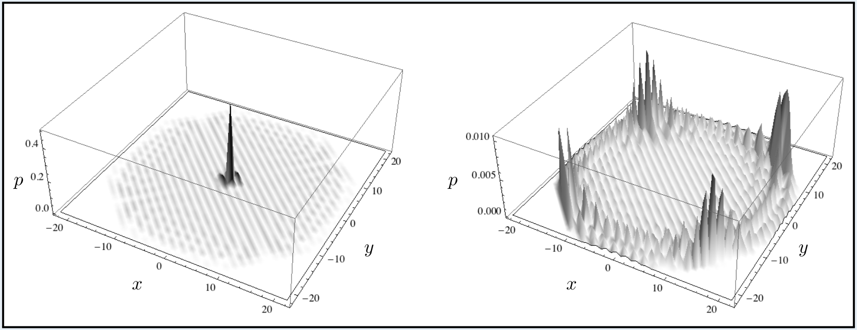
- Time evolution in momentum picture:

$$\tilde{U}(k,l) = \begin{pmatrix} e^{-ik} & 0 & 0 & 0 \\ 0 & e^{-il} & 0 & 0 \\ 0 & 0 & e^{il} & 0 \\ 0 & 0 & 0 & e^{ik} \end{pmatrix} \cdot \exp i \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12}^* & c_{22} & c_{23} & c_{24} \\ c_{31}^* & c_{23}^* & c_{33} & c_{34} \\ c_{41}^* & c_{24}^* & c_{34}^* & c_{44} \end{pmatrix}$$

- $w(k,l) =$???
- Numerically: possibly zero measure (like the 1D case)
- Known solutions: Grover coin [J. Phys. A: Math. Gen. **35**, 2745 (2002)]
  Watabe coin [Phys. Rev. A **77**, 062331 (2008)]

Grover trapping

- Grover coin: $G = \begin{pmatrix} e^{-ik} & 0 & 0 & 0 \\ 0 & e^{-il} & 0 & 0 \\ 0 & 0 & e^{il} & 0 \\ 0 & 0 & 0 & e^{ik} \end{pmatrix}$
- Trapping for all localized initial state. Exception: $\frac{1}{2}\left(|L\rangle - |D\rangle - |U\rangle + |R\rangle\right)$



A novel trapping coin class

Formula

- $C = (\text{C-phase}\ \ -\varphi)\,(U_1 \otimes U_2)\,(\text{C-phase}\ \varphi)(\text{Swap})\,; \quad U_1, U_2 \in SU(2)$

Derivation

- V_1V_2 and V_2V_1 are unitary equivalent $SU(2)$
- $(V_1V_2) \otimes (V_2V_1) \Rightarrow$ eigenvalue $1^{(2)}$
- $(V_1 \otimes V_2)(\text{Swap})(V_1 \otimes V_2)(\text{Swap}) \Rightarrow (V_1 \otimes V_2)(\text{Swap}) \Rightarrow$ eigenvalues ± 1
- $V_i \overset{!}{=} \tilde{D}_i U_i \Rightarrow$ quantum walk propagator
- Controlled-phase gate can be added without losing constant eigenvalues

Relation to known trapping walks

- Largest trapping set so far (7 real parameters)
- Grover coin is included
- Watabe coin is included
- Flip-flop Grover (quantum search) is included

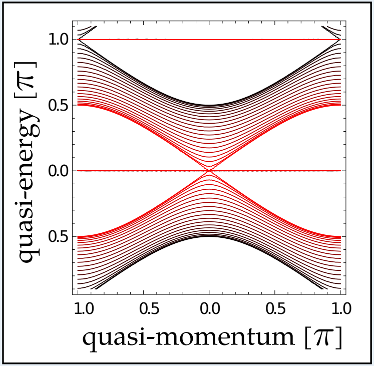
Spectrum

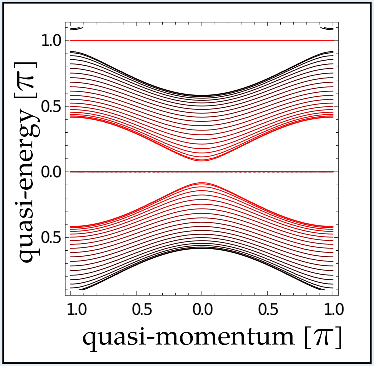
Formula of the coin operator

- $C = (\text{C-phase } -\varphi)\,(U_1 \otimes U_2)\,(\text{C-phase } \varphi)(\text{Swap})\,; \quad U_1, U_2 \in SU(2)$
- $U_k = \begin{pmatrix} e^{-i\alpha_k}\cos\delta_k & -e^{-i\beta_k}\sin\delta_k \\ e^{i\beta_k}\sin\delta_k & e^{i\alpha_k}\cos\delta_k \end{pmatrix}$

Spectrum of the walk (~ split step!)

- $\delta_{1(2)}$ tune the continuity

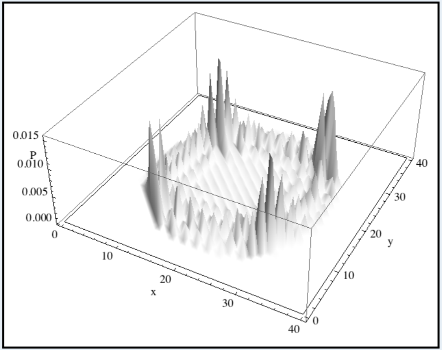


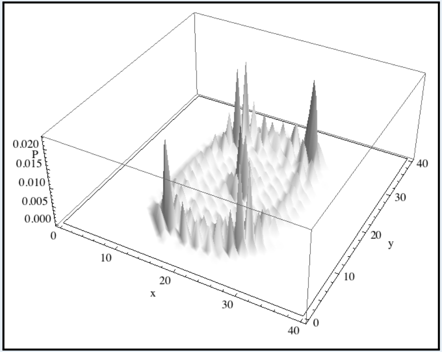


Strong trapping

Strong trapping

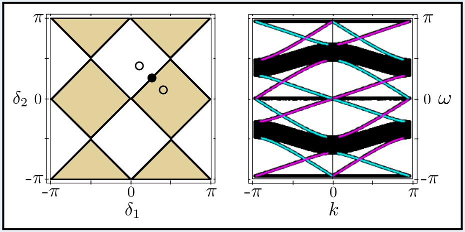
- Gapless spectrum ⇒ Localized initial states can escape using a single well chosen internal state
- Gapped spectrum ⇒ all localized initial states are trapped: **strong trapping**





Topological phases, quantum search

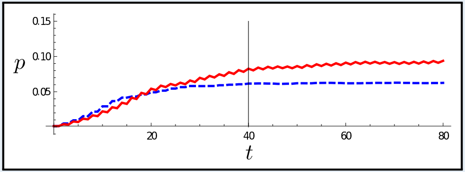
Topological phases



- Open question: topological invariants

Quantum search [AKR2005] on the non-square tori

- Higher probability to find the marked vertex than the Grover walk



- Open question: optimization (analytical)

Summary - Outlook

Summary: The strong trapping walk [PRA 91, 022308 (2015)]

- Class with 7 real parameter (the largest so far!)
- New feature: strong trapping
- Topological phases are demonstrated (Open question: invariants?)
- Faster search on non-square tori (Open question: optimization?)
- Spectral correspondence with the 2D split-step walk
- Multidimensional generalization? (Search on hypercuboids?)

Outlook: The general trapping problem of quantum walks

- Goal: Classification of quantum walks (2D). All trapping coins?
- Problem: "Brute force" analysis or numerics does not work
- Systematic method for building trapping coin classes ("induction")
- Easy-to-adapt to other quantum walk definitions (Lazy, higher dimensions)
- Complete* solution: 4 type of coins for 2D quantum walks (+ permutations)
  * 90% work in progress
- B. Kollár, A. Gilyén, T. Kiss and I. Jex, *in preparation*
- Future plans: All recurrent quantum walks?

Acknowledgements

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