Positivity of linear maps under tensor powers

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Wigner's Theorem

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PPT criterion

If $(\Theta_A \otimes \mathrm{id}_B)(\rho_{AB}) =: \rho_{AB}^\Gamma \not\geq 0$, then ρ_{AB} is entangled.

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recall: • $\mathcal{T}$ *positive* if $\mathcal{T}(X) \geq 0 \;\; \forall X \geq 0$

• $\mathcal{T}$ *completely positive* if $(\mathcal{T} \otimes \mathrm{id})(X_{AB}) \geq 0 \quad \forall X_{AB} \geq 0$

$[\Rightarrow \;\; (\mathcal{T}_A \otimes \mathcal{T}_B \otimes \cdots \otimes \mathcal{T}_N)(X_{AB\cdots N}) \geq 0 \quad \forall X_{AB\cdots N} \geq 0\,]$

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Quantum capacity bound (Holevo/Werner)

For any quantum channel $\mathcal{T}$:

$$\mathcal{Q}(\mathcal{T}) \leq \log \|\Theta \circ \mathcal{T}\|_\diamond .$$

$\to$ If $\Theta \circ \mathcal{T}$ CP, then $\mathcal{Q}(\mathcal{T}) = 0$.

“Tensor-stable positive maps”

Here: behaviour under tensor powers

Main definition: Let $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be a linear map.

(a) $\mathcal{P}$ is called n**-tensor-stable positive** for some $n \in \mathbb{N}$ if $\mathcal{P}^{\otimes n} : \mathcal{M}_{d_1^n} \to \mathcal{M}_{d_2^n}$ is a positive map.

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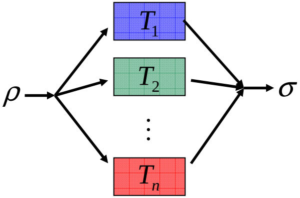
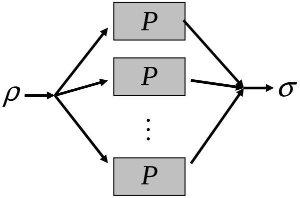
$\mathcal{T}_1, \ldots, \mathcal{T}_n$ compl. pos.



$\mathcal{P}$ n-tensor-stable positive



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- the **composition** of two: $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$
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$\to$ **"trivial" tensor-stable positive maps** $:= \{\text{CP}, \text{coCP}\}$.

Why tensor-stable positive maps?

1. Applications:

- generalize capacity bound $\mathcal{Q}(\mathcal{T}) \leq \log \|\Theta \circ \mathcal{T}\|_\diamond$
- strong converse rates for $\mathcal{Q}(\mathcal{T})$ and $\mathcal{Q}_2(\mathcal{T})$
- link to ∞-*locally entanglement annihilating channels* $\mathcal{T}$

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2. Existence question:

- Do (n-)tensor-stable positive $\mathcal{P} \notin \{\text{CP}, \text{coCP}\}$ exist?
- relation to NPT bound entanglement

Quantum capacity bound

$\mathcal{P} : \mathcal{M}_{d_2} \to \mathcal{M}_{d_2}$ tensor-stable positive, bijective (unital), *not* CP.

Theorem: For any quantum channel $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$:

$$\mathcal{Q}(\mathcal{T}) \leq \frac{\log\left(\|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond \|\mathcal{P}^*(I)\|_\infty\right)}{\log \|\mathcal{P}^*\|_\diamond} \log d_2 \,.$$

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- diamond norm: $\|\mathcal{M}\|_\diamond := \sup_{d,X} \frac{\|(\mathcal{M} \otimes \mathrm{id}_d)(X)\|_1}{\|X\|_1}$

  $\to$ efficiently computable!

- for $\mathcal{P} = \Theta_{d_2}$: Holevo-Werner bound:

  $$\mathcal{Q}(\mathcal{T}) \leq \log \|\Theta \circ \mathcal{T}\|_\diamond$$

- **but:** no improvement for $\mathcal{P} = \Theta \circ \mathcal{S} \in$ coCP

  $\to$ need **non-trivial** $\mathcal{P}$

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Proof: Suppose $\frac{1}{2}\|\text{id}_{d_2}^{\otimes n} - \mathcal{D} \circ \mathcal{T}^{\otimes m} \circ \mathcal{E}\|_\diamond \leq \varepsilon$.

Key fact: $(\Theta\mathcal{P}^*\Theta)^{\otimes n} \circ \mathcal{D} \circ \mathcal{P}^{\otimes m}$ is CP if $\mathcal{P}$ ts-positive.

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Choi matrix: $\left((\Theta\mathcal{P}^*\Theta)^{\otimes n} \circ \mathcal{D} \circ \mathcal{P}^{\otimes m} \otimes \mathrm{id}\right)(\Omega)$

$$= \left((\Theta\mathcal{P}^*\Theta)^{\otimes n} \circ \mathcal{D} \circ \mathrm{id} \otimes (\Theta\mathcal{P}^*\Theta)^{\otimes m}\right)(\Omega)$$

$$= \underbrace{(\Theta\mathcal{P}^*\Theta)^{\otimes(n+m)}}_{\text{positive map}} \circ \underbrace{(\mathcal{D} \otimes \mathrm{id})(\Omega)}_{\geq 0} \geq 0$$

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....

$$\Rightarrow \underbrace{\frac{n}{m}}_{\limsup_{n,m}\cdots=\mathcal{Q}(\mathcal{T})} \leq \frac{\log\left(\|\mathcal{P}^{-1}\circ\mathcal{T}\|_\diamond \|\mathcal{P}^*(I)\|_\infty\right)}{\log\|\mathcal{P}^*\|_\diamond} - \underbrace{\frac{\log(1-2\varepsilon)}{m\log\|\mathcal{P}^*\|_\diamond}}_{\text{OK if } \varepsilon<1/2} \quad \square$$

Strong converse rate for quantum capacity

Track fidelity $F(\Omega, \mathcal{D} \circ \mathcal{T}^{\otimes m} \circ \mathcal{E}(\Omega))$ of the max entangled state:

$$\mathcal{Q}(\mathcal{T}) \le \underbrace{\frac{\log\left(\|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond \|\mathcal{P}^*(I)\|_\infty\right)}{\log \|\mathcal{P}^*\|_\diamond} \log d_2}_{\text{previous slide: capacity bound}}$$

$$\le \underbrace{\frac{\log\left(\|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond \|\mathcal{P}^*(I)\|_\infty\right)}{\log \|(\mathcal{P}^* \otimes \mathrm{id})(\Omega)\|_1} \log d_2}_{\text{strong converse rate for } \mathcal{Q}}$$

Link to entanglement-annihilating channels

T is called ∞-*locally entanglement annihilating* iff

$$\mathcal{T}^{\otimes k}(\rho) \text{ fully separable } \forall k \, \forall \rho.$$

OK if $\mathcal{T}$ entanglement-breaking, i.e. $(\mathcal{T} \otimes \mathrm{id})(\rho)$ separable $\forall \rho$.

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Theorem

If such $\mathcal{T}$ exists which is *not* entanglement-breaking,
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Proof: (a) $\mathcal{T}$ not EB $\Rightarrow$ find positive $\mathcal{S}$ s.th.

$$\mathcal{S} \circ \mathcal{T} \ \text{ is not CP.}$$

(b) define $\mathcal{P} := (\mathcal{S} \circ \mathcal{T}) \otimes (\Theta \circ \mathcal{S} \circ \mathcal{T})$.

$$\Rightarrow \quad \mathcal{P}^{\otimes n}(\rho) = \big(\mathcal{S} \otimes (\Theta \circ \mathcal{S})\big)^{\otimes n} \circ \underbrace{\mathcal{T}^{\otimes (2n)}(\rho)}_{\text{separable!}} \geq 0. \qquad \square$$

But wait! Is there any **non-trivial** ts-positive $\mathcal{P}$?

(recall: "non-trivial" $:\Leftrightarrow \ \mathcal{P} \notin \{\text{CP}, \text{coCP}\}$)

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[construction via unextendible product bases]

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Corollary: All tensor-stable positive maps $\mathcal{P} : \mathcal{M}_2 \to \mathcal{M}_d$ or $\mathcal{P} : \mathcal{M}_d \to \mathcal{M}_2$ are trivial: $\mathcal{P} \in \{\text{CP}, \text{coCP}\}$

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$\mathcal{S}_{LOCC}(\text{Choi}_{\mathcal{P}})$ is NPT (Werner) state.

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(b) contradiction: If $\mathcal{S}_{LOCC}\,(\text{Choi}_{\mathcal{P}})$ were distillable, then

$$\Big\|\Omega_{d_1} - \underbrace{\big(\mathcal{S}_n \circ \mathcal{S}_{LOCC}^{\otimes n}\big)(\text{Choi}_{\mathcal{P}}^{\otimes n})}_{\sum_i (\mathcal{D}_i \circ \mathcal{P}^{\otimes n} \circ \mathcal{E}_i \otimes \mathrm{id})(\Omega)}\Big\|_1 \to 0 \qquad \text{as } n \to \infty\,.$$

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$$\Rightarrow \text{Choi}_{\mathcal{P}} = (\text{id} \otimes \mathcal{P})(\Omega_{d_1}) \approx \Big(\sum_i \mathcal{D}_i \circ \mathcal{P}^{\otimes n} \circ \mathcal{E}_i \otimes \mathcal{P} \Big)(\Omega)$$

$$= \sum_i (\mathcal{D}_i \otimes \text{id}) \circ \underbrace{\mathcal{P}^{\otimes (n+1)}}_{\geq 0} \circ (\mathcal{E}_i \otimes \text{id})(\Omega) \ \geq \ 0.$$

Conclusion

- tensor-stable positive maps: $\mathcal{P}^{\otimes n} \geq 0 \ \ \forall n$
- gives new capacity bounds, ...
- **open question:** Does such $\mathcal{P} \notin \{\mathrm{CP}, \mathrm{coCP}\}$ exist?
  - $\rightarrow$ mathem. characterization of transposition
  - $\rightarrow$ existence of NPT bound entanglement
  - $\rightarrow$ existence of entanglement-annihilating channels
- existence question reduced to $\mathcal{W}_p \otimes (\Theta \circ \mathcal{W}_p)$
  - $\rightarrow$ improve construction?
- unital $\mathcal{T}$ is n-ts positive $\;\Leftrightarrow\; \|\mathcal{T}^{\otimes n}\|_{\infty \rightarrow \infty} = 1.$

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Thanks!

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Want: $\text{Choi}_{\mathcal{P}} := C_{AB} \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2}$ should satisfy

$$\big(\langle \Psi_A| \otimes \langle \Phi_B|\big)\, C_{AB}^{\otimes n} \big(|\Psi_A\rangle \otimes |\Phi_B\rangle\big) \;\geq\; 0 \qquad \forall |\Psi_A\rangle, |\Phi_B\rangle \qquad (1)$$

("block-positive"), but $C_{AB} \not\geq 0$, $C_{AB}^{\Gamma} \not\geq 0$.

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("block-positive"), but $C_{AB} \not\geq 0$, $C_{AB}^{\Gamma} \not\geq 0$.

Construction: $C_{AB} = P_{AB} - \varepsilon I$ where $P \geq 0$ and

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Now: Non-trivial ts-positive $\Rightarrow$ NPT-BE

(2) If $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ non-trivial tensor-stable positive, then there exist **NPT bound entangled states** in $\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1}$.

First: Key lemma connecting to *tensor-stable positivity*

Second: Standard distillation on *block-positive* operators

Key lemma: relation to NPT-BE

Let $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be $(n+1)$-tensor-stable positive. Then:

$$\frac{d_{\mathsf{CP}}(\mathcal{P})}{\|\mathcal{P}\|_\diamond} \leq \inf_{\mathcal{S}\in\mathsf{LOCC}} \left\|\Omega_{d_1} - \mathcal{S}(\mathsf{Choi}_{\mathcal{P}}^{\otimes n})\right\|_1 ,$$

where $d_{\mathsf{CP}}(\mathcal{P}) := \frac{1}{2}\|\mathsf{Choi}_{\mathcal{P}}\|_1 - \frac{1}{2}\mathrm{Tr}[\mathsf{Choi}_{\mathcal{P}}] = \mathrm{dist}(\mathcal{P}, \mathsf{CP\ cone})$.

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Idea: Have $\Omega_{d_1} \approx \mathcal{S}(\mathsf{Choi}_{\mathcal{P}}^{\otimes n}) = \underbrace{\sum_i (\mathcal{D}_i \otimes \mathcal{E}_i)}_{\mathcal{S}} \underbrace{(\mathcal{P}^{\otimes n} \otimes \mathrm{id})(\Omega)}_{\mathsf{Choi}_{\mathcal{P}}^{\otimes n}}$:

$$\Rightarrow \quad \mathrm{id}_{d_1} \approx \sum_i \mathcal{D}_i \circ \mathcal{P}^{\otimes n} \circ \widetilde{\mathcal{E}}_i .$$

$$\Rightarrow \quad \mathrm{id}_{d_1} \otimes \mathcal{P} \approx \Big(\sum_i \mathcal{D}_i \circ \mathcal{P}^{\otimes n} \circ \widetilde{\mathcal{E}}_i \Big) \otimes \mathcal{P}$$

$$= \sum_i (\mathcal{D}_i \otimes \mathrm{id}) \circ \mathcal{P}^{\otimes (n+1)} \circ (\widetilde{\mathcal{E}}_i \otimes \mathrm{id}) \geq 0.$$

If approximation good, then $\mathcal{P}$ is “close” to CP. □