Conditions for optimal input states for discrimination of quantum channels

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Formulation of the problem

$\Phi_1, \ldots, \Phi_n \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, assume that a channel Φ is one of $\Phi_1, \ldots, \Phi_n$, with prior probabilities $\lambda_1, \ldots, \lambda_n$. For meausrement given by triple $(\mathcal{H}_0, \rho, M)$ the average success probability is

$$p(M, \rho) = \sum_i \lambda_i \mathrm{Tr}\, M_i (\Phi_i \otimes id)(\rho)$$

The task is to maximize its value over all triples $(\mathcal{H}_0, \rho, M)$. In terms of PPOVM:

$$p(M, \rho) = \sum_i \lambda_i \mathrm{Tr}\, C(\Phi_i) F_i,$$

$C(\Phi_i)$ - Choi matrix of Φ_i, F_i - PPOVM,

$$\sum_i F_i = I \otimes \sigma.$$

SDP formulation

Maximization of the success probability can be written as a problem of semidefinite programming:

$$\begin{aligned}\max_{F\in B(\mathbb{C}^n\otimes\mathcal{K}\otimes\mathcal{H})} \quad & \mathrm{Tr}\, CF \\ \text{s.t.} \quad \mathrm{Tr}\,(I\otimes X_i)F &= 0, \quad i=1,\dots,m, \\ F &\geq 0, \\ \mathrm{Tr}\, F &= \dim(\mathcal{K}).\end{aligned}$$

Here $X_1,\dots,X_m$ is any basis of the (real) linear subspace

$$\mathcal{L} := \{X = X^* \in B(\mathcal{K}\otimes\mathcal{H}), \mathrm{Tr}_{\mathcal{K}} X = 0\}.$$

$C = \sum_{i=1}^n |i\rangle\langle i| \otimes \lambda_i C(\Phi_i)$,
$F = \sum_{i=1}^n |i\rangle\langle i| \otimes F_i \in B(\mathbb{C}^n\otimes\mathcal{K}\otimes\mathcal{H})$, then

$$\mathrm{Tr}\, CF = \sum_i \lambda_i \mathrm{Tr}\, C(\Phi_i)F_i.$$

Zero duality gap

Theorem

Let $\hat{F}$ be a process POVM. Then $\hat{F}$ is optimal if and only if there is some $\lambda_0 \geq 0$ and some $\Phi_0 \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, such that for all i,

$$\begin{aligned} \lambda_i C(\Phi_i) &\leq \lambda_0 C(\Phi_0), \\ (\lambda_0 C(\Phi_0) - \lambda_i C(\Phi_i))\hat{F}_i &= 0, \qquad \forall i. \end{aligned}$$

Moreover, in this case, the maximal success probability is

$$\mathrm{Tr}\, \hat{F} C = \max_F \mathrm{Tr}\, F C = \min_{\Phi \in \mathcal{C}(\mathcal{H},\mathcal{K})} \min\{\lambda, \lambda_i C(\Phi_i) \leq \lambda C(\Phi),\ \forall i\}.$$

Proof.

$F' = \frac{1}{n \dim(\mathcal{H})} I$ is a primal feasible plan, by Slater's condition duality gap is zero. □

Corollary of zero duality gap

Corollary

Let $\rho \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$ *be a pure state such that* $\mathrm{Tr}_1 \rho =: \sigma$ *is invertible. Then a measurement scheme* $(\mathcal{H}, \rho, M)$ *is optimal if and only if*

(i) $Z := \sum_i \lambda_i M_i (\Phi_i \otimes id)(\rho)$ *majorizes* $\lambda_i (\Phi_i \otimes id)(\rho)$ *for all* i *and*

(ii) $\mathrm{Tr}_{\mathcal{K}} Z \propto \sigma$.

Error estimate

Theorem

Let $Z = \sum_i \lambda_i C(\Phi_i) M_i$*,* $p_{MEI} = \dim(\mathcal{H})^{-1} \mathrm{Tr}\, Z$*, where* M *is an optimal POVM for discrimination of the ensemble* $\{\lambda_i, \dim(\mathcal{H})^{-1} C(\Phi_i)\}$*. Let* $\|\cdot\|$ *denote the operator norm. Then the optimal success probability* p_{opt} *satisfies*

$$p_{MEI} \leq p_{opt} \leq \|\mathrm{Tr}_{\mathcal{K}} Z\|.$$

Maximally entangled input state

Only two channels, $\Delta_\lambda = \lambda C_1 - (1-\lambda)C_2$, $\lambda \in (0,1)$.
Optimal POVM: $\{M, I - M\}$, M is the projection onto $\mathrm{supp}(\Delta_\lambda^+)$.

Corollary

An optimal measurement scheme $(\mathcal{H}, \rho, M)$ with a pure maximally entangled input state ρ exists if and only if the Choi operators satisfy

$$\mathrm{Tr}_{\mathcal{K}}|\Delta_\lambda| \propto I. \qquad \text{(MEI)}$$

Proof.

Such a scheme exists if and only if $\mathrm{Tr}_{\mathcal{K}} Z \propto I$, equivalently, $\mathrm{Tr}_{\mathcal{K}}(\Delta_\lambda)_+ \propto I$. Since we always have $\mathrm{Tr}_{\mathcal{K}}\Delta_\lambda \propto I$ and

$$(\Delta_\lambda)_+ = \frac{1}{2}(\Delta_\lambda + |\Delta_\lambda|),$$

the condition can be rewritten as stated. □

Applications - covariant channels

Proposition

Let Φ_1, Φ_2 *be covariant channels, i.e.* $\Phi_i(U_g \rho U_g^*) = V_g \Phi_i(\rho) V_g^*$*, then the condition* (MEI) *is satisfied if the representation* $g \mapsto U_g$ *is irreducible.*

Proof.

Follows from

$$U_g^t(\mathrm{Tr}\,_{\mathcal{K}}|\Delta_\lambda|)\bar{U}_g = \mathrm{Tr}\,_{\mathcal{K}}|\Delta_\lambda|.$$

□

If U is reducible, then we must have

$$\mathrm{Tr}_{\mathcal{K}}|\Delta_\lambda| = \sum_i k_i P_i.$$

Let $t_i = \mathrm{Tr}\, P_i$ then $p_{\mathrm{MEI}} = \frac{1}{2}(1 + \dim(\mathcal{H})^{-1} \sum_i t_i k_i)$ and we have

$$p_{\mathrm{opt}} \leq \frac{1}{2}\left(1 + \max_i k_i\right).$$

Applications - qubit channels

$\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, let $\Gamma(X) = (\mathrm{Tr}\, X)I - X^t$ be the Werner-Holevo channel, where X^t - transpose map with respect to the basis $|0\rangle, |1\rangle$. For a self-adjoint $X \in B(\mathbb{C}^2)$

$$\Gamma(X) = X^t \qquad \Leftrightarrow \qquad X \propto I.$$

Proposition

For a pair of qubit channels, the condition (MEI) *holds if and only if*

$$\mathrm{Tr}_{\mathcal{K}}|\Delta_\lambda + ((2\lambda - 1)I - \Phi_\lambda(I)) \otimes I| = \mathrm{Tr}_{\mathcal{K}}|\Delta_\lambda|.$$

where $\Phi_\lambda = \lambda\Phi_1 - (1-\lambda)\Phi_2$.

Proof.

By direct calculation of $\Gamma(\mathrm{Tr}_{\mathcal{K}}|\Delta_\lambda|)$. □

Note: MEI holds if $\Phi_\lambda(I) = (2\lambda - 1)I$.

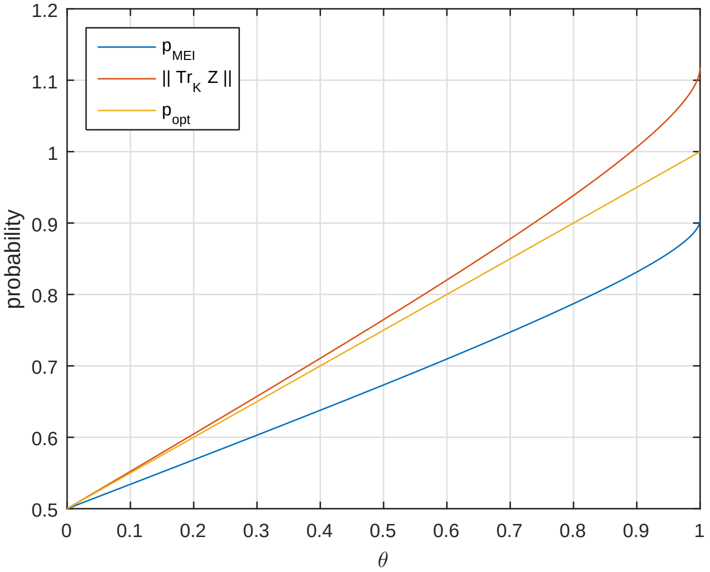
Example - qubit channels

The task of discrimination of identity channel ϕ_{id} and amplitude damping channel ϕ_{AD}, $\lambda = \frac{1}{2}$. Φ_{AD} is represented by Kraus operators A_θ, B_θ, defined as

$$A_\theta = |0\rangle\langle 0| + \sqrt{1-\theta}|1\rangle\langle 1|, \qquad B_\theta = \sqrt{\theta}|0\rangle\langle 1|,$$

where $\theta \in [0,1]$.
Solving for p_{opt}: $p_{opt} = \frac{1}{2}(1 + \|\lambda\Phi_1 - (1-\lambda)\Phi_2\|_\diamond)$, p_{opt} was obtained numerically by CVX and Matlab.

Example - qubit channels



Applications - unitary channels

Let $U, V \in \mathcal{U}(\mathcal{H})$ and let $\Phi_1 = Ad_U$, $\Phi_2 = Ad_V$ be the corresponding unitary channels.

Proposition

Put $W = V^* U$ and let $\lambda \in (0, 1)$. Then (MEI) *holds if and only if either $\mathrm{Tr}\, W = 0$ or W has at most two different eigenvalues, each of the same multiplicity.*

Proof.

By diagonalizing $D_\lambda := \lambda P_\psi - (1 - \lambda) P_\phi$. □

Note: $\mathrm{Tr}\, W = 0$ then the measured states are orthogonal. If dim($\mathcal{H}$) is odd, (MEI) holds iff $\mathrm{Tr}\, W = \mathrm{Tr}\, V^* U = \langle \psi, \phi \rangle = 0$.

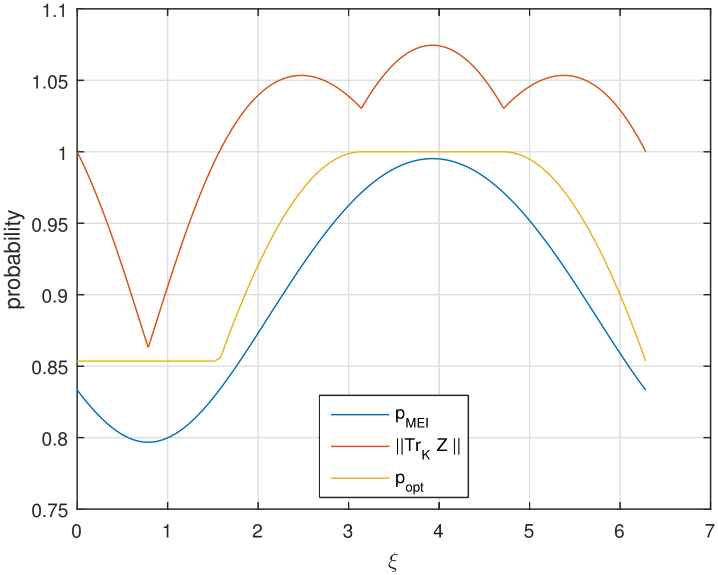
Examples - unitary channels

Let $\lambda = 1/2$ and $\dim(\mathcal{H}) = 3$. Without loss of generality, we assume that we discriminate the identity channel against a unitaries Ad_{W_i}, $i = 1, 2$.

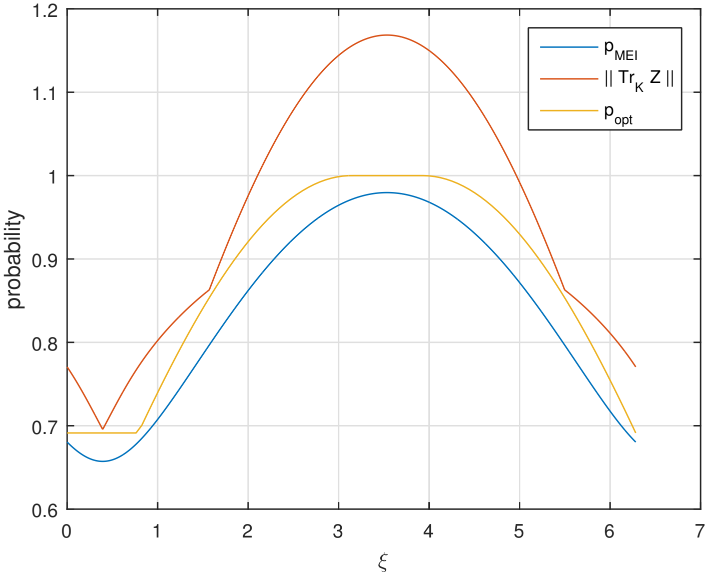
$$W_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & e^{i\xi} \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & 0 & e^{i\xi} \end{pmatrix},$$

for $\xi \in [0, 2\pi]$.
p_{MEI}, $\|\mathrm{Tr}_{\mathcal{K}} Z\|$ are obtained by direct calculation, p_{opt} by numerical solution of the SDP problem for the diamond norm.

Examples - unitary channels



Examples - unitary channels



Applications - simple projective measurements

Simple projective measurement is given by a sharp POVM $M = (P_1, \dots, P_m)$. The corresponding channel $\Phi_P : B(H) \to B(\mathbb{C}^m)$ is defined as $A \mapsto \sum_i (\operatorname{Tr} P_i A) |i\rangle\langle i|$.

Applications - simple projective measurements

$\{P_{\xi_i}\}$, $\{P_{\eta_i}\}$ - sharp POVMs.
$[W_{ij}] = [\langle \xi_i, \eta_j \rangle]$ - unitary matrix.

Proposition

Assume that $P_{\xi_j} = P_{\eta_j}$ for some j. Then the condition MEI is satisfied if and only if $M = N$.

Proposition

Let $c_i = (1 - |\langle \xi_i, \eta_i \rangle|^2)^{1/2}$ and assume that $c_i = c$ for all i. Then

- (i) *If $c \neq 1$ and $\dim(\mathcal{H})$ is odd, then (MEI) cannot be satisfied.*
- (ii) *If $c \neq 1$ and $\dim(\mathcal{H})$ is even, then (MEI) is satisfied if and only if $W = \sqrt{1 - c^2} I + G$, where G is some suitable hollow matrix satisfying $G^* = -G$ and $G^2 = -c^2 I$.*
- (iii) *If $c = 1$, then (MEI) always holds.*

Example - simple projective measurements

$\lambda = \frac{1}{2}$, $\dim(\mathcal{H}) \geq 3$,

$$|\eta_1\rangle = \frac{1}{\sqrt{2}}(|\xi_1\rangle + |\xi_2\rangle)$$
$$|\eta_2\rangle = \frac{1}{\sqrt{2}}(|\xi_1\rangle - |\xi_2\rangle)$$
$$|\eta_j\rangle = |\xi_j\rangle \qquad j \geq 3.$$

Maximally entangled input state is not optimal, because $|\eta_j\rangle = |\xi_j\rangle$ for $j \geq 3$. By direct calculations we get

$$p_{\mathrm{MEI}} = \frac{1}{2} + \frac{1}{\sqrt{2}\dim(\mathcal{H})},$$
$$p_{\mathrm{opt}} = \frac{2+\sqrt{2}}{4} \approx 0.8535\ldots$$
$$\|\mathrm{Tr}_{\mathcal{K}} Z\| = \frac{2+\sqrt{2}}{4} \approx 0.8535\ldots$$

Thank you for your attention.