

Lower and upper bounds on classical capacity of nonunitary channels

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CEQIP

Smolence

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Plan

1. Classical capacity of a quantum channel
2. Difficulties with nonunital channels
3. Quantum Sinkhorn's theorem
4. Bounds on capacity
4. Improvement of bounds

Classical capacity

Set of words

Classical capacity

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$$|\mathcal{S}| = N$$

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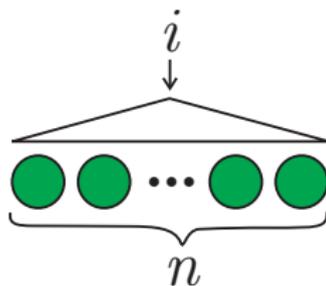
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$$|\mathcal{S}| = N$$

$$\text{"wine"} = i \in \{1, \dots, N\}$$

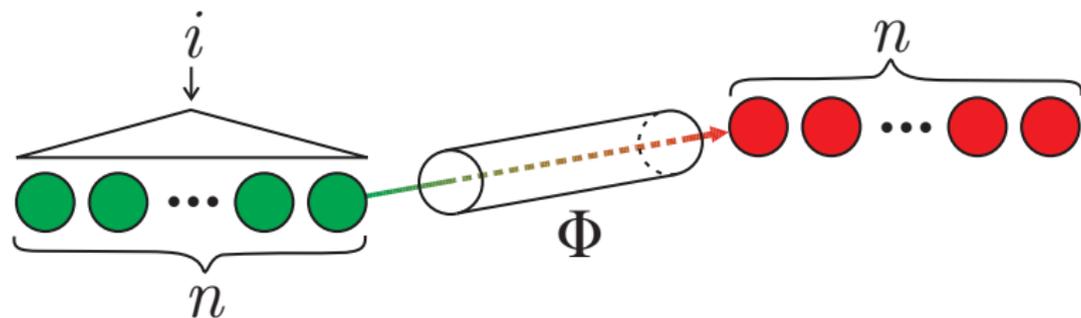
Classical capacity



Encoder: $i \rightarrow \rho_i^{(n)}$

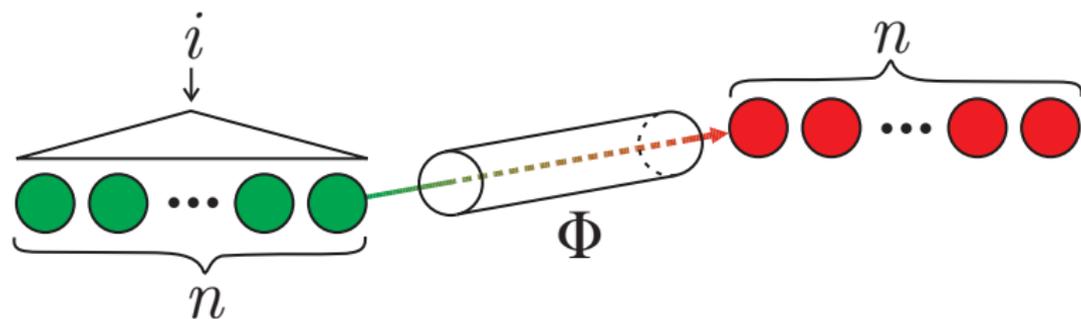
n is the number of qubits

Classical capacity



Quantum channel Φ is a CPT map for individual qubit

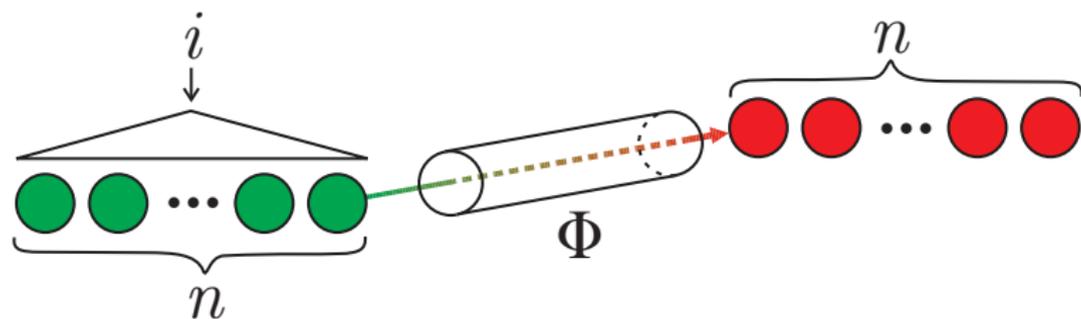
Classical capacity



Quantum channel Φ is a CPT map for individual qubit

Map $\Phi^{\otimes n}$ for n qubits

Classical capacity

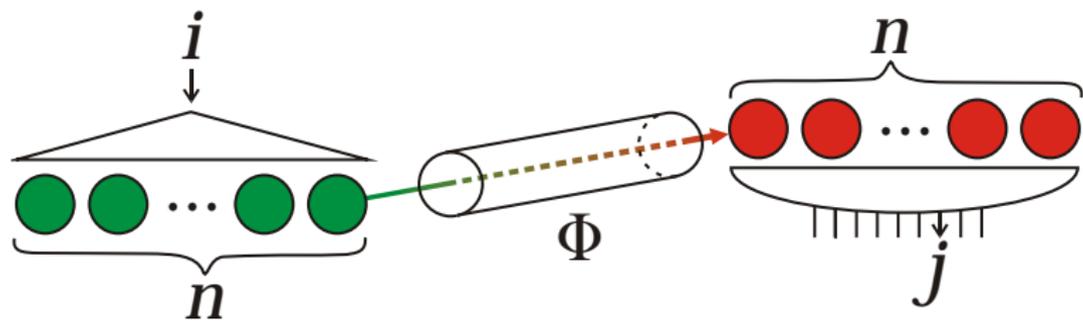


Quantum channel Φ is a CPT map for individual qubit

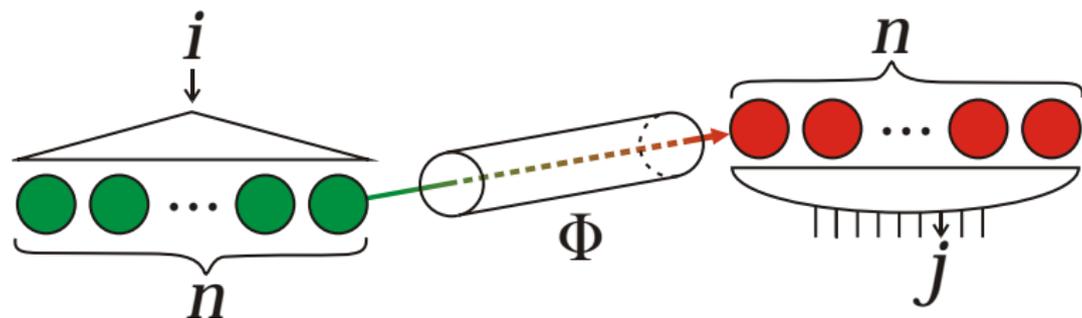
Map $\Phi^{\otimes n}$ for n qubits

The output state of n qubits is $\Phi^{\otimes n}[\rho_i^{(n)}]$

Classical capacity

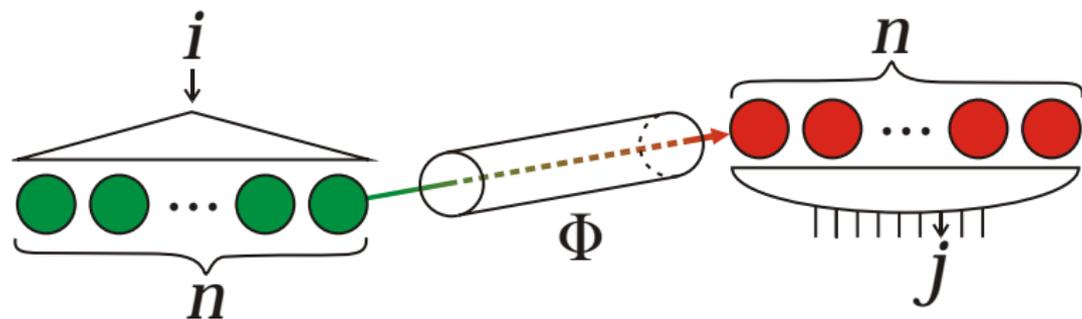


Classical capacity



Decoder: POVM, which assigns a positive-semidefinite operator $M_j^{(n)}$ (acting on 2^n -dimensional Hilbert space) to each observed outcome $j \in \{1, \dots, N\}$

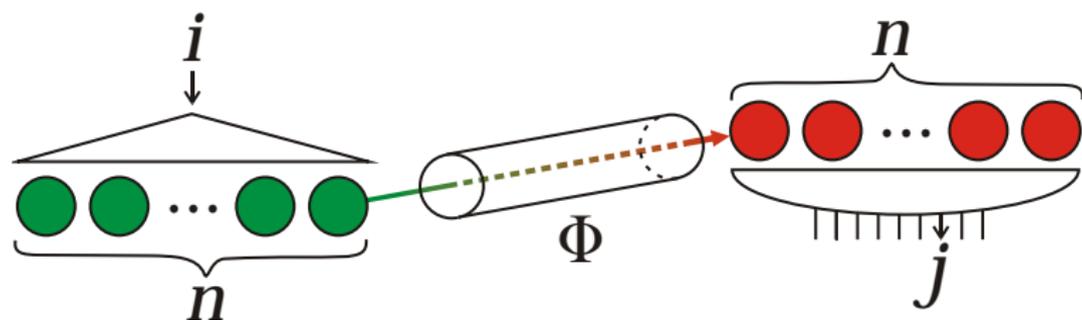
Classical capacity



Decoder: POVM, which assigns a positive-semidefinite operator $M_j^{(n)}$ (acting on 2^n -dimensional Hilbert space) to each observed outcome $j \in \{1, \dots, N\}$

$$p^{(n)}(j|i) = \text{tr}[\rho_i^{(n)} M_j^{(n)}]$$

Classical capacity

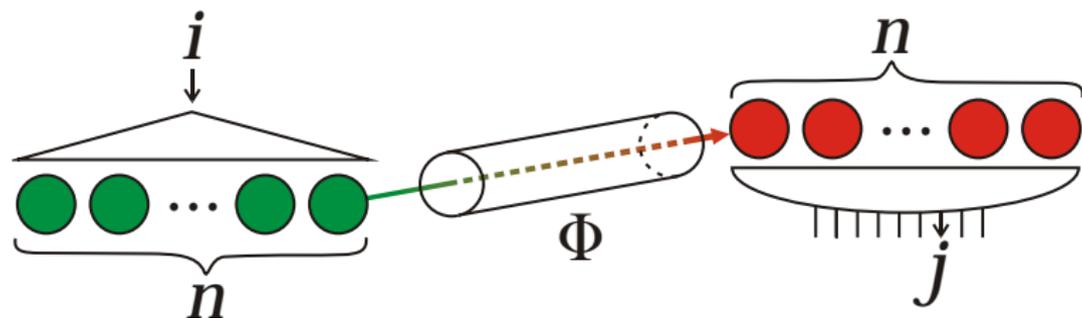


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Condition $\sum_{j=1}^N M_j^{(n)} = I$ guarantees $\sum_{j=1}^N p^{(n)}(j|i) = 1$.

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$$p_{\text{err}}(n, N) = \max_{j=1, \dots, N} \left(1 - p^{(n)}(j|j) \right)$$

Classical capacity

R is called an achievable rate of information transmission if

$$\lim_{n \rightarrow \infty} p_{\text{err}}(n, 2^{nR}) = 0$$

Classical capacity:

$$C(\Phi) = \sup \left\{ R : \lim_{n \rightarrow \infty} p_{\text{err}}(n, 2^{nR}) = 0 \right\}$$

Holevo¹–Schumacher–Westmoreland² theorem:

$$C(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} C_{\chi}(\Phi^{\otimes n})$$

$$C_{\chi}(\Psi) = \sup_{\{p_k, \rho_k\}} \left[S \left(\sum_k p_k \Psi[\rho_k] \right) - \sum_k p_k S(\Psi[\rho_k]) \right]$$

$$S(\rho) = -\text{tr}(\rho \log_2 \rho)$$

¹A. S. Holevo, IEEE Trans. Inf. Theory 44, 269 (1998).

²B. Schumacher, M. Westmoreland, Phys. Rev. A 56, 131 (1997).

Classical capacity

Additivity property

$$C_{\chi}(\Phi^{\otimes n}) = nC_{\chi}(\Phi)$$

holds for a limited classes of channels only (depolarizing channels³, entanglement breaking channels⁴, unital qubit channels⁵).

³C. King, IEEE Trans. Inf. Theory **49**, 221 (2003).

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Υ is unital if $\Upsilon[I] = I$

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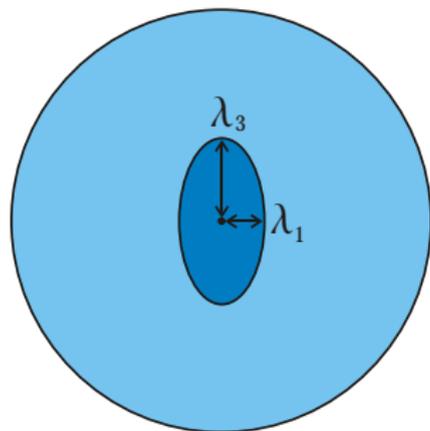
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Unital qubit channel:

$$\Upsilon[X] = \frac{1}{2} \left(\text{tr}[X]I + \sum_{k=1}^3 \text{tr}[X\sigma_k] \lambda_k \sigma_k \right)$$



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Classical capacity of unital qubit channels

$$C(\Upsilon) = C_{\chi}(\Upsilon) = 1 - h\left(\frac{1}{2}\left(1 - \max_{i=1,2,3} |\lambda_i|\right)\right)$$

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

Classical capacity of unital qubit channels

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Optimal encodings
and decodings are known!

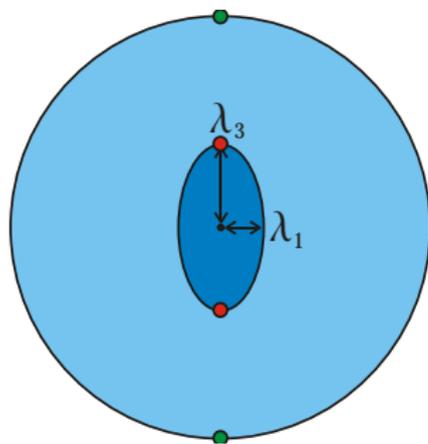
Message

$i \rightarrow$ binary form $0,1,0,0,1,1,\dots$

$$\rho_i^{(n)} = |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \dots$$

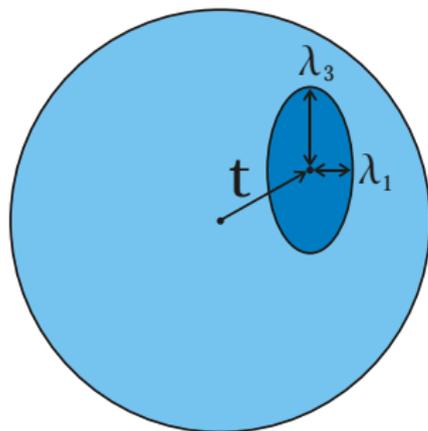
$$M_j^{(n)} = \sum_{x: g(x)=j} \bigotimes_{k=1}^n M_{x_k}^{(1)},$$

$$M_{x_k}^{(1)} \in \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$$



Classical capacity of nonunitary qubit channels

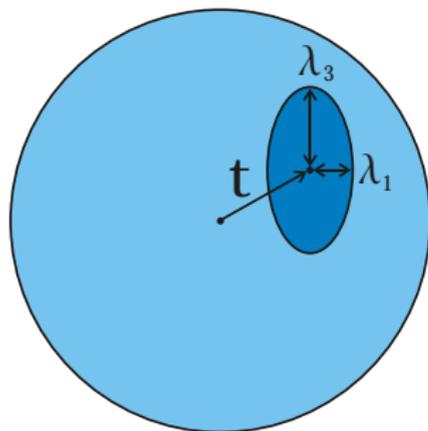
$$\Phi[I] \neq I$$



Classical capacity of nonunital qubit channels

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What is the capacity
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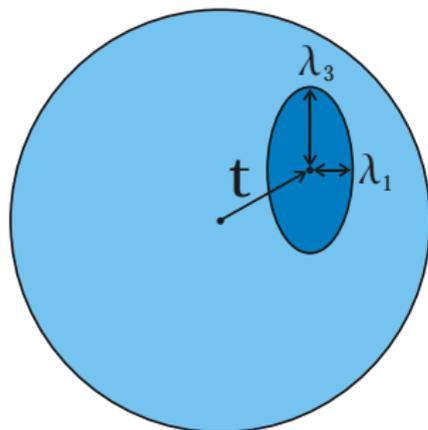


Classical capacity of nonunital qubit channels

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What is the capacity
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Nobody knows



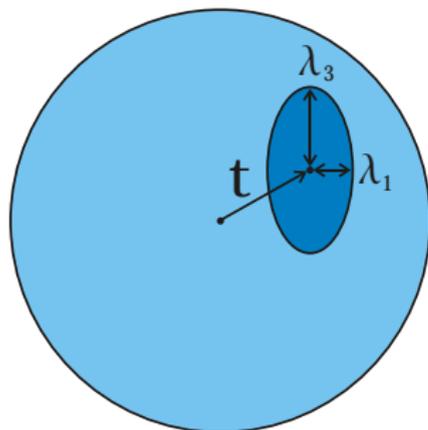
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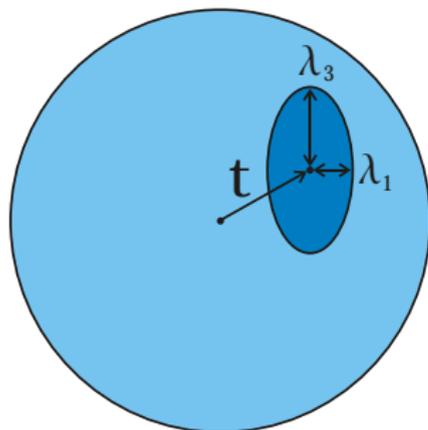


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Bounds:

- ▶ X. Wang, W. Xie, R. Duan, Semidefinite programming strong converse bounds for classical capacity, *IEEE Trans. Inf. Theory* **64**, 640 (2018).
- ▶ F. Leditzky, E. Kaur, N. Datta, M. M. Wilde, Approaches for approximate additivity of the Holevo information of quantum channels, *Phys. Rev. A* **97**, 012332 (2018).

Quantum Sinkhorn's theorem

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Many thanks to David Reeb, who focused my attention on that.

⁶G. Aubrun, S. J. Szarek, Two proofs of Størmer's theorem, arXiv:1512.03293 (2015)

Quantum Sinkhorn's theorem



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Let A and B be operators acting on \mathcal{H}_d . Denote

$$\Phi_A[X] = AXA^\dagger, \quad \Phi_B[X] = BXB^\dagger$$

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Let A and B be operators acting on \mathcal{H}_d . Denote

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Theorem ⁽⁶⁾

Let $\Phi : \mathcal{B}(\mathcal{H}_d) \mapsto \mathcal{B}(\mathcal{H}_d)$ be a linear map which belongs to the interior of the cone of positive maps. Then there exist positive-definite operators A and B such that $\Upsilon = \Phi_A \circ \Phi \circ \Phi_B$ is unital and trace preserving.

⁶G. Aubrun, S. J. Szarek, Two proofs of Størmer's theorem, arXiv:1512.03293 (2015)

Bounds on capacity

Proposition⁷. Suppose Φ is a channel such that $\Psi = \Phi_A \circ \Phi \circ \Phi_B$ is a channel too. Then $C(\Phi) \geq C(\Psi) - 2 \log_2(\|A\| \|B\|)$.

⁷S. N. Filippov, arXiv:1802.00646 [quant-ph]

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Proof. Let $\{\rho_i^{(n)}, M_i^{(n)}\}_{i=1}^N$ be the optimal code of size $N = 2^{nR_\Psi}$ for the composite channel $\Psi^{\otimes n}$ s.t. $\lim_{n \rightarrow \infty} p_{\text{err}} \Psi(n, 2^{nR_\Psi}) = 0$.

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Modified input states:

$$\tilde{\varrho}_i^{(n)} = \frac{B^{\otimes n} \varrho_i^{(n)} (B^\dagger)^{\otimes n}}{\text{tr}[B^{\otimes n} \varrho_i^{(n)} (B^\dagger)^{\otimes n}]}.$$

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Modified positive operator-valued measure $\{j \rightarrow \tilde{M}_j^{(n)}\}_{j=0}^N$:

$$\tilde{M}_0^{(n)} = I - \sum_{j=1}^N \tilde{M}_j^{(n)}, \quad \tilde{M}_j^{(n)} = \frac{(A^\dagger)^{\otimes n} M_j^{(n)} A^{\otimes n}}{\|A\|^{2n}}, \quad j = 1, \dots, N,$$

$\|X\| = \|X\|_\infty = \max_{\psi: \langle \psi | \psi \rangle = 1} \langle \psi | X^\dagger X | \psi \rangle$ is the operator norm.

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Bounds on capacity

Using the modified code, let each qubit be transmitted through the channel Φ . Then the probability to observe outcome $j \neq 0$ provided input message i equals

$$\tilde{p}^{(n)}(j|i) = \text{tr} \left[\tilde{\rho}_i^{(n)} \tilde{M}_j^{(n)} \right] = \frac{\text{tr} \left\{ A^{\otimes n} \Phi^{\otimes n} \left[B^{\otimes n} \rho_i^{(n)} (B^\dagger)^{\otimes n} \right] (A^\dagger)^{\otimes n} M_j^{(n)} \right\}}{\text{tr} [B^{\otimes n} \rho_i^{(n)} (B^\dagger)^{\otimes n}] \|A\|^{2n}}$$

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Since $\Phi_A \circ \Phi \circ \Phi_B = \Psi$, we get

$$\tilde{p}^{(n)}(j|i) = \frac{\text{tr} \left\{ \Psi^{\otimes n} [\rho_i^{(n)}] M_j^{(n)} \right\}}{\text{tr} [B^{\otimes n} \rho_i^{(n)} (B^\dagger)^{\otimes n}] \|A\|^{2n}} = \frac{p^{(n)}(j|i)}{\text{tr} [B^{\otimes n} \rho_i^{(n)} (B^\dagger)^{\otimes n}] \|A\|^{2n}},$$

where $p^{(n)}(j|i)$ is the probability to get outcome $j \in \{1, \dots, N\}$ for the input message $i \in \{1, \dots, N\}$ in the original optimal protocol for channel $\Psi^{\otimes n}$.

Bounds on capacity

Observation of the outcome $j = 0$ in the modified protocol would be treated as unsuccessful event, whereas observation of the outcome $j \in \{1, \dots, N\}$ leads to a successful identification of the message because $p^{(n)}(j|i) \rightarrow \delta_{ij}$ if $n \rightarrow \infty$.

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The probability to observe nonzero outcome j equals

$$P^{(n)} = \sum_{j=1}^N \tilde{p}^{(n)}(j|i) = \frac{1}{\text{tr}[B^{\otimes n} \rho_i^{(n)} (B^\dagger)^{\otimes n}] \|A\|^{2n}} \geq \frac{1}{(\|A\| \|B\|)^{2n}}$$

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One can transmit information in the case of successful events $j \neq 0$, the average number of successfully transmitted messages \tilde{N} is

$$\tilde{N} = P^{(n)} N = P^{(n)} 2^{nR_\Psi} \geq 2^{n(R_\Psi - 2 \log_2(\|A\| \|B\|))}$$

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Therefore, the considered protocol enables one to achieve the rate

$$\tilde{R} \geq R_\Psi - 2 \log_2(\|A\| \|B\|)$$

Bounds on capacity

If $R_\Psi \leq C(\Psi)$ and one observes the successful event ($j \neq 0$), then the maximum error probability in the modified protocol

$$\begin{aligned}\tilde{p}_{\text{err}}(n, \tilde{N}) &= \max_{j=1, \dots, N} \left(1 - \frac{\tilde{p}^{(n)}(j|j)}{P^{(n)}} \right) \\ &= \max_{j=1, \dots, N} \left(1 - p^{(n)}(j|j) \right) \xrightarrow{n \rightarrow \infty} 0.\end{aligned}$$

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Taking supremum on both sides of $\tilde{R} \geq R_\Psi - 2 \log_2(\|A\| \|B\|)$ with requirement $\lim_{n \rightarrow \infty} \tilde{p}_{\text{err}}(n, \tilde{N}) = 0$, we get

$$C(\Phi) \geq C(\Psi) - 2 \log_2(\|A\| \|B\|)$$

Q.E.D.

Bounds on capacity

$$\Upsilon = \Phi_A \circ \Phi \circ \Phi_B$$

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Corollary (⁸)

Let Φ be a unital qubit channel belonging to the interior of positive qubit maps, then there exist positive definite operators A and B acting on \mathcal{H}_2 such that the map $\Upsilon = \Phi_A \circ \Phi \circ \Phi_B$ is a unital channel and

$$C(\Upsilon) - 2 \log_2(\|A\| \|B\|) \leq C(\Phi) \leq C(\Upsilon) + 2 \log_2(\|A^{-1}\| \|B^{-1}\|).$$

⁸S. N. Filippov, arXiv:1802.00646 [quant-ph]

4-parameter nonunital qubit channels

Nonunital qubit channel

$$\Phi[X] = \frac{1}{2} \left(\text{tr}[X](I + t_3\sigma_3) + \sum_{j=1}^3 \lambda_j \text{tr}[\sigma_j \rho] \sigma_j \right) \text{ with} \\ |t_3| + |\lambda_3| < 1$$

4-parameter nonunital qubit channels

Nonunital qubit channel

$$\Phi[X] = \frac{1}{2} \left(\text{tr}[X](I + t_3\sigma_3) + \sum_{j=1}^3 \lambda_j \text{tr}[\sigma_j \rho] \sigma_j \right) \text{ with} \\ |t_3| + |\lambda_3| < 1$$

$$A = \text{diag} \left(\sqrt[4]{(1-t_3)^2 - \lambda_3^2}, \sqrt[4]{(1+t_3)^2 - \lambda_3^2} \right)$$

$$B = \frac{\sqrt{2} \left(4 - \left(\sqrt[4]{(1-t_3)^2 - \lambda_3^2} - \sqrt[4]{(1+t_3)^2 - \lambda_3^2} \right)^2 \right)^{-1/2}}{\sqrt[4]{(1-t_3)^2 - \lambda_3^2} \sqrt[4]{(1+t_3)^2 - \lambda_3^2}} \\ \times \text{diag} \left(\sqrt{\left((1+t_3 - \lambda_3) \sqrt[4]{(1-t_3)^2 - \lambda_3^2} + (1-t_3 + \lambda_3) \sqrt[4]{(1+t_3)^2 - \lambda_3^2} \right)^2}, \right. \\ \left. \sqrt{\left((1+t_3 + \lambda_3) \sqrt[4]{(1-t_3)^2 - \lambda_3^2} + (1-t_3 - \lambda_3) \sqrt[4]{(1+t_3)^2 - \lambda_3^2} \right)^2} \right)$$

4-parameter nonunital qubit channels

Nonunital qubit channel

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$$C(\Upsilon) - 2 \log_2(\|A\| \|B\|) \leq C(\Phi) \leq C(\Upsilon) + 2 \log_2(\|A^{-1}\| \|B^{-1}\|).$$

4-parameter nonunital qubit channels

Unital qubit channel Υ has parameters⁹

$$\begin{aligned}\tilde{\lambda}_1 &= \frac{2\lambda_1}{\sqrt{(1+\lambda_3)^2 - t_3^2} + \sqrt{(1-\lambda_3)^2 - t_3^2}}, \\ \tilde{\lambda}_2 &= \frac{2\lambda_2}{\sqrt{(1+\lambda_3)^2 - t_3^2} + \sqrt{(1-\lambda_3)^2 - t_3^2}}, \\ \tilde{\lambda}_3 &= \frac{4\lambda_3}{\left(\sqrt{(1+\lambda_3)^2 - t_3^2} + \sqrt{(1-\lambda_3)^2 - t_3^2}\right)^2}.\end{aligned}$$

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Example

Following Ref. ¹⁰, consider a one-parameter qubit channel

$$\Phi_{\text{mix}} = p\mathcal{A}_p + (1 - p)\mathcal{D}_p,$$

where $0 \leq p \leq 1$,

$\mathcal{A}_p[X] = K_1 X K_1^\dagger + K_2 X K_2^\dagger$ is the qubit amplitude damping channel with $K_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|$ and $K_2 = \sqrt{p}|0\rangle\langle 1|$,

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$$\mathcal{D}_p[X] = (1 - p)X + \frac{p}{3}(\sigma_x X \sigma_x + \sigma_y X \sigma_y + \sigma_z X \sigma_z).$$

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Φ_{mix} is a partial case of the 4-parameter channel discussed before:

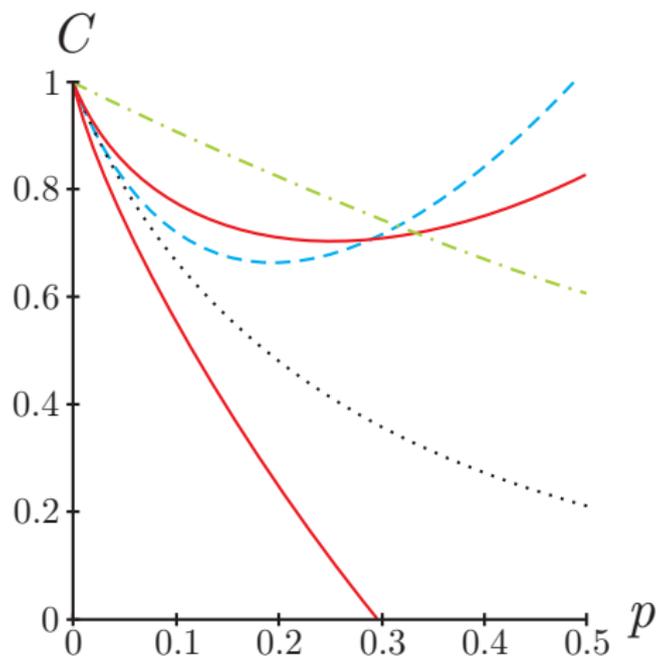
$$\lambda_1 = \lambda_2 = p\sqrt{1-p} + (1-p) \left(1 - \frac{4p}{3}\right)$$

$$\lambda_3 = (1-p) \left(1 - \frac{p}{3}\right)$$

$$t_3 = p^2.$$

¹⁰F. Leditzky, E. Kaur, N. Datta, M. M. Wilde, Phys. Rev. A 97, 012332 (2018)

Example



- ▶ X. Wang, W. Xie, R. Duan, *IEEE Trans. Inf. Theory* **64**, 640 (2018), upper bound
- ▶ F. Leditzky, E. Kaur, N. Datta, M. M. Wilde, *Phys. Rev. A* **97**, 012332 (2018), upper bound
- ▶ S. N. Filippov, [arXiv:1802.00646 \[quant-ph\]](https://arxiv.org/abs/1802.00646), upper and lower bounds
- ▶ $C_{\chi}(\Phi_{\text{mix}})$, lower bound

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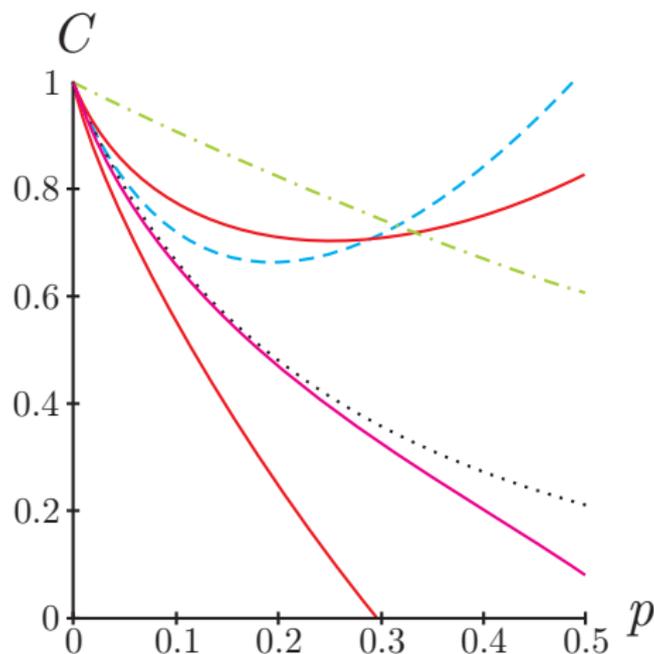
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Since $\varrho_i^{(n)} = \bigotimes_{k=1}^n \varrho_{i_k}$ and we know B explicitly, we can replace $\|B\|^2$ by $\max_{i,k} \|B\varrho_{i_k}B^\dagger\|$

This approach works for improvement of lower bound.

Improvement of bounds



- ▶ X. Wang, W. Xie, R. Duan, *IEEE Trans. Inf. Theory* **64**, 640 (2018), upper bound
- ▶ F. Leditzky, E. Kaur, N. Datta, M. M. Wilde, *Phys. Rev. A* **97**, 012332 (2018), upper bound
- ▶ S. N. Filippov, [arXiv:1802.00646 \[quant-ph\]](https://arxiv.org/abs/1802.00646), upper and lower bounds
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- ▶ improved lower bound

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- ▶ The bounds can be further improved.

Thank you for listening!

Quantum Sinkhorn's theorem

Proof.

$$\Upsilon[I] = A\Phi[B^2]A = I \iff (\Phi[B^2])^{-1} = A^2$$

$$\Upsilon^\dagger[I] = B\Phi^\dagger[A^2]B = I \iff (\Phi^\dagger[A^2])^{-1} = B^2$$

$$\left(\Phi\left[(\Phi^\dagger[S])^{-1}\right]\right)^{-1} = S$$

$$A = S^{1/2}$$

$$B = (\Phi^\dagger[S])^{-1/2}$$

Quantum Sinkhorn's theorem

S is a fixed point of the map $F[X] = \left(\Phi \left[(\Phi^\dagger[X])^{-1} \right] \right)^{-1}$

$$f[X] = \frac{F[X]}{\text{tr}[F[X]]}$$

By Brouwer's fixed-point theorem there exists a density operator ρ such that $f[\rho] = \rho$ and hence $F[\rho] = \alpha\rho$, where $\alpha = \text{tr}[F[\rho]] > 0$. If we choose $A = \rho^{1/2}$ and $B = (\Phi^\dagger[\rho])^{-1/2}$, then Υ is trace preserving and satisfies $\Phi[I] = \alpha I$. Therefore, if $\alpha = 1$, then ρ is a fixed point of F that we needed to conclude the proof.

Quantum Sinkhorn's theorem

Alternative discussions of the relation $\Upsilon = \Phi_A \circ \Phi \circ \Phi_B$:

- ▶ L. Gurvits, Classical complexity and quantum entanglement, J. Comput. System Sci. **69**, 448–484 (2004).

Definition. For a positive map $T : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_d)$, we define its capacity as $\text{Cap}(T) = \inf\{\det T[X] \mid X > 0, \det X = 1\}$.

Theorem

There exist non-singular matrices C_1, C_2 such that $\Phi_{C_1} \circ T \circ \Phi_{C_2}^\dagger$ is doubly stochastic iff the infimum in Definition above is strictly positive and attained. Moreover, if $\text{Cap}(T) = \det T[C]$ where $C > 0, \det C = 1$, then $\Phi_{(T[C])^{-1/2}} \circ T \circ \Phi_{C^{1/2}}$ is doubly stochastic.

- ▶ T. T. Georgiou, M. Pavon, Positive contraction mappings for classical and quantum Schrödinger systems, J. Math. Phys. **56**, 033301 (2015).

For so-called positivity improving CPT maps with the property $\Phi^\dagger[\rho] > 0$ for all ρ .